STA447/2006 Midterm, February 8, 2018

(135 minutes; 5 questions; 3 pages; total points = 45)

[SOLUTIONS]

1. Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$, and transition probabilities $p_{11} = p_{12} = 1/2$, $p_{21} = 1/3$, $p_{22} = 2/3$, $p_{32} = 1/7$, $p_{33} = 2/7$, $p_{34} = 4/7$, $p_{44} = 1$.

(a) **[2]** Compute $p_{32}^{(2)}$. (You do <u>not</u> need to simplify the final fraction.)

Solution. $p_{32}^{(2)} = \sum_{k \in S} p_{3k} p_{k2} = p_{31}p_{12} + p_{32}p_{22} + p_{33}p_{32} + p_{34}p_{42} = (0)(1/2) + (1/7)(2/3) + (2/7)(1/7) + (4/7)(0) = 2/21 + 2/49$.

(b) **[2]** Determine whether or not $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$. [Hint: perhaps let $C = \{1, 2\}$.]

Solution. *The subset $C = \{1, 2\}$ is <u>closed</u> since $p_{ij} = 0$ for $i \in C$ and $j \notin C$. Furthermore, the Markov chain restricted to C is irreducible (since it's possible to go $1 \to 2 \to 1$), and C is finite. Hence, by the Finite State Space Theorem, we must have $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$.*

(c) **[4]** Compute (with explanation) f_{32}.

Solution. *Here $f_{32} = \sum_{n=1}^{\infty} \mathbf{P}_3[\text{first hit 2 at time } n] = \sum_{n=1}^{\infty} (2/7)^{n-1}(1/7) = (1/7)/(1 - (2/7)) = (1/7)/(5/7) = 1/5$. Or, alternatively, $f_{32} = \mathbf{P}_3[\text{hit 2 when we first leave 3}] = \mathbf{P}_3[\text{hit 2} \mid \text{leave 3}] = (1/7)/((1/7) + (4/7)) = 1/5$. Or, alternatively, by the F-Expansion, $f_{32} = p_{32} + p_{31} f_{12} + p_{33} f_{32} + p_{34} f_{42} = (1/7) + 0 + (2/7)f_{32} + 0$, so $(5/7)f_{32} = 1/7$, so $f_{32} = (1/7)/(5/7) = 1/5$.*

2. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) **[3]** The chain is irreducible, with period 3, and has a stationary distribution.

Solution. *Possible. For example, let $S = \{1, 2, 3\}$, with $p_{12} = p_{23} = p_{31} = 1$ (and $p_{ij} = 0$ otherwise). Then the chain is irreducible (since it can get from $1 \to 2 \to 3 \to 1$), and periodic with period 3 (since it only returns to each i in multiples of three steps). Furthermore the chain is doubly stochastic, so if $\pi_1 = \pi_2 = \pi_3 = 1/3$, then π is a stationarity distribution.*

(b) **[3]** There is $k \in S$ having period 2, and $\ell \in S$ having period 4.

Solution. *Possible. For example, let $S = \{1, 2, 3, 4, 5, 6\}$, with $p_{12} = p_{21} = 1$, and with $p_{34} = p_{45} = p_{56} = p_{63} = 1$. Then state $k = 1$ has period 2 since it only returns in multiples of 2 steps, and state $\ell = 3$ has period 4 since it only returns in multiples of 4 steps. (Of course, this chain is not irreducible; for irreducible chains, all states must have the same period.)*

(c) **[3]** The chain has a stationary distribution π, and $0 < p_{ij} < 1$ for all $i, j \in S$, but the chain is <u>not</u> reversible with respect to π.

Solution. *Possible. For example, let $S = \{1, 2, 3\}$, with $p_{12} = p_{23} = p_{31} = 1/3$, and $p_{21} = p_{32} = p_{13} = 1/2$, and $p_{11} = p_{22} = p_{33} = 1/6$. Then $0 < p_{ij} < 1$ for all $i, j \in S$ (yes, even when $i = j$). Next, let $\pi_1 = \pi_2 = \pi_3 = 1/3$, so π is a probability distribution on S. Then $\pi_1 p_{12} = (1/3)(1/3) \neq (1/3)(1/2) = \pi_2 p_{21}$, so the chain is not reversible with respect to π. On the other hand, for any $j \in S$, we have $\sum_i \pi_i p_{ij} = (1/3)(1/3 + 1/2 + 1/6) = 1/3 = \pi_j$. (Or, alternatively, $\sum_i p_{ij} = 1/3 + 1/2 + 1/6 = 1$, so the chain is doubly stochastic.) Hence, π is a stationary distribution.*

(d) **[3]** The chain is irreducible, and there are distinct states $i, j, k, \ell \in S$ such that $f_{ij} < 1$, and $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$.

Solution. *Not possible. If the chain is irreducible, and $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$, then by the Stronger Recurrence Theorem, we must have $f_{ij} = 1$ for all i and j.*

(e) **[3]** The chain is irreducible, and there are are distinct states $i, j, k \in S$ with $p_{ij} > 0$, $p_{jk}^{(2)} > 0$, and $p_{ki}^{(3)} > 0$, and state i is <u>periodic</u> with period equal to an <u>odd</u> number.

Solution. *Possible. For example, let $S = \{1, 2, 3, 4, 5, 6\}$, with $p_{12} = p_{15} = 1/2$, and $p_{23} = p_{34} = p_{45} = p_{56} = p_{61} = 1$, with $p_{ij} = 0$ o.w. Let $i = 1$, and $j = 2$, and $k = 4$. Then $p_{ij} = p_{12} = 1/2 > 0$, and $p_{jk}^{(2)} = p_{23}p_{34} = 1(1) = 1 > 0$, and $p_{ki}^{(3)} = p_{45}p_{56}p_{61} = 1(1)(1) = 1 > 0$, but state i has period 3 (which is odd) since from i the chain can return to i in three steps ($1 \to 5 \to 6 \to 1$) or six steps ($1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1$), and $gcd(3, 6) = 3$.*

(f) **[3]** There are distinct states $i, j, k \in S$ with $f_{ij} = 1/2$, $f_{jk} = 1/3$, and $f_{ik} = 1/10$.

Solution. *Not possible. One way to eventually get from i to k, is to first eventually get from i to j, and then eventually get from j to k. This means we must have $f_{ik} \geq f_{ij}\, f_{jk} = (1/2)(1/3) = 1/6$, so we cannot have $f_{ik} = 1/10$.*

3. Consider the Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{12} = p_{32} = 1$, $p_{21} = 1/4$, and $p_{23} = 3/4$. Let $\pi_1 = 1/8$, $\pi_2 = 1/2$, and $\pi_3 = 3/8$.

(a) **[3]** Verify that the chain is reversible with respect to π.

Solution. *Here $\pi_1 p_{12} = (1/8)(1) = (1/2)(1/4) = \pi_2 p_{21}$, and $\pi_1 p_{13} = (1/8)(0) = (3/8)(0) = \pi_3 p_{31}$, and $\pi_3 p_{32} = (3/8)(1) = (1/2)(3/4) = \pi_2 p_{23}$, so $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$, so the chain is reversible with respect to π.*

(b) **[6]** Determine (with explanation) which of the following statements are true and which are false: (i) $\lim_{n\to\infty} p_{11}^{(n)} = 1/8$. (ii) $\lim_{n\to\infty} \frac{1}{2}[p_{11}^{(n)} + p_{11}^{(n+1)}] = 1/8$. (iii) $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} p_{11}^{(\ell)} = 1/8$.

Solution. *Here π is stationary by part (a), and the chain is irreducible since it can go $1 \to 2 \to 3 \to 2 \to 1$, but the chain has period 2 since it always moves from odd to even or from even to odd. Hence, $p_{11}^{(n)} = 0$ whenever n is odd, so we do <u>not</u> have $\lim_{n\to\infty} p_{11}^{(n)} = 1/8$.*

But by the Periodic Convergence Theorem, we do *still have* $\lim_{n\to\infty} \frac{1}{2}[p_{11}^{(n)} + p_{11}^{(n+1)}] = \pi_1 = 1/8$, *and by the Periodic Convergence Corollary we also have* $\lim_{n\to\infty} \frac{1}{n}\sum_{\ell=1}^{n} p_{11}^{(\ell)} = \pi_1 = 1/8$. *So, in summary, (i) does* not *hold, but (ii) and (iii)* do *hold.*

4. **[5]** Consider the undirected graph with vertex set $V = \{1,2,3,4\}$, and an undirected edge (of weight 1) between each of the following four pairs of vertices (and no other edges): (1,2), (2,3), (3,4), (2,4). Let $\{p_{ij}\}_{i,j\in V}$ be the transition probabilities for random walk on this graph. Compute (with full explanation) $\lim_{n\to\infty} p_{12}^{(n)}$, or prove this limit does not exist.

Solution. *The graph is connected (since we can get from* $1 \to 2 \to 3 \to 4$ *and back), so the walk is irreducible. Also, the walk is aperiodic since e.g. we can get from 2 to 2 in 2 steps by* $2 \to 3 \to 2$, *or in 3 steps by* $2 \to 3 \to 4 \to 2$, *and* $gcd(2,3) = 1$. *Here* $Z = \sum_u d(u) = 2|E| = 2(4) = 8 < \infty$. *Hence, as shown in class, if* $\pi_u = d(u)/Z$, *then the walk is reversible with respect to* π, *so* π *is a stationary distribution. Also* $d(2) = 3$, *because there are three edges from the vertex 2. Hence, by the Graph Convergence Theorem,* $\lim_{n\to\infty} p_{12}^{(n)} = \pi_2 = d(2)/Z = 3/8$.

5. **[5]** Let $\{p_{ij}\}$ be the transition probabilities for an irreducible Markov chain with state space S. Let $i,j,k,\ell \in S$. Suppose $\lim_{n\to\infty} p_{k\ell}^{(n)} = 0$. Prove that $\lim_{n\to\infty} p_{ij}^{(n)} = 0$.
[Hint: since $k \to i$ and $j \to \ell$, there are times $r, s \in \mathbf{N}$ with $p_{ki}^{(r)} > 0$ and $p_{j\ell}^{(s)} > 0$.]

Solution. *Find* $r, s \in \mathbf{N}$ *with* $p_{ki}^{(r)} > 0$ *and* $p_{j\ell}^{(s)} > 0$. *Then by Chapman-Kolmogorov,* $p_{k\ell}^{(r+n+s)} \ge p_{ki}^{(r)} p_{ij}^{(n)} p_{j\ell}^{(s)}$, *so* $p_{ij}^{(n)} \le p_{k\ell}^{(r+n+s)} \big/ (p_{ki}^{(r)} p_{j\ell}^{(s)})$. *But* $\lim_{n\to\infty} \left[p_{k\ell}^{(r+n+s)} \big/ (p_{ki}^{(r)} p_{j\ell}^{(s)})\right] = 0$. *Also* $p_{ij}^{(n)} \ge 0$. *So,* $p_{ij}^{(n)}$ *is "sandwiched" between 0 and a sequence converging to 0. Hence, by the Sandwich Theorem (or, Squeeze Theorem) from Calculus, we must have* $\lim_{n\to\infty} p_{ij}^{(n)} = 0$. *(Or, less formally but not* quite *correct, since* $p_{ij}^{(n)}$ *is non-negative and is* $\le$ *something going to zero, therefore* $p_{ij}^{(n)}$ *must also go to zero.)*

[END OF EXAMINATION: total points = 45]