STA447/2006 Midterm, February 8, 2018

(135 minutes; 5 questions; 6 pages; total points = 45)

FAMILY NAME: ______________ **GIVEN NAME(S):** ______________

STUDENT #: ______________ **SIGNATURE:** ______________

Class (circle one): **STA447** **STA2006**

Do not open this booklet until told to do so. Answer all questions.

You <u>may</u> use results from class. Aids allowed: NONE.

Point values for each question are indicated [in square brackets].

You should <u>explain</u> all of your solutions clearly.

You may continue on the back of the page if necessary (write "OVER").

DO NOT WRITE BELOW THIS LINE.

Question	Score
1(a)	/2
1(b)	/2
1(c)	/4
2(a)	/3
2(b)	/3
2(c)	/3
2(d)	/3
2(e)	/3
2(f)	/3

Question	Score
3(a)	/3
3(b)	/6
4	/5
5	/5
TOTAL:	/45

1. Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$, and transition probabilities $p_{11} = p_{12} = 1/2$, $p_{21} = 1/3$, $p_{22} = 2/3$, $p_{32} = 1/7$, $p_{33} = 2/7$, $p_{34} = 4/7$, $p_{44} = 1$.

(a) [**2**] Compute $p_{32}^{(2)}$. (You do <u>not</u> need to simplify the final fraction.)

(b) [**2**] Determine whether or not $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$. [Hint: perhaps let $C = \{1, 2\}$.]

(c) [**4**] Compute (with explanation) f_{32}.

2. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j\in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) [**3**] The chain is irreducible, with period 3, and has a stationary distribution.

(b) [**3**] There is $k \in S$ having period 2, and $\ell \in S$ having period 4.

(c) [**3**] The chain has a stationary distribution π, and $0 < p_{ij} < 1$ for all $i, j \in S$, but the chain is <u>not</u> reversible with respect to π.

2. (cont'd)

(d) [**3**] The chain is irreducible, and there are distinct states $i, j, k, \ell \in S$ such that $f_{ij} < 1$, and $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$.

(e) [**3**] The chain is irreducible, and there are are distinct states $i, j, k \in S$ with $p_{ij} > 0$, $p_{jk}^{(2)} > 0$, and $p_{ki}^{(3)} > 0$, and state i is periodic with period equal to an odd number.

(f) [**3**] There are distinct states $i, j, k \in S$ with $f_{ij} = 1/2$, $f_{jk} = 1/3$, and $f_{ik} = 1/10$.

3. Consider the Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{12} = p_{32} = 1$, $p_{21} = 1/4$, and $p_{23} = 3/4$. Let $\pi_1 = 1/8$, $\pi_2 = 1/2$, and $\pi_3 = 3/8$.

(a) [**3**] Verify that the chain is reversible with respect to π.

(b) [**6**] Determine (with explanation) which of the following statements are true and which are false: (i) $\lim_{n\to\infty} p_{11}^{(n)} = 1/8$. (ii) $\lim_{n\to\infty} \frac{1}{2}[p_{11}^{(n)} + p_{11}^{(n+1)}] = 1/8$. (iii) $\lim_{n\to\infty} \frac{1}{n}\sum_{\ell=1}^{n} p_{11}^{(\ell)} = 1/8$.

4. **[5]** Consider the undirected graph with vertex set $V = \{1, 2, 3, 4\}$, and an undirected edge (of weight 1) between each of the following four pairs of vertices (and no other edges): (1,2), (2,3), (3,4), (2,4). Let $\{p_{ij}\}_{i,j\in V}$ be the transition probabilities for random walk on this graph. Compute (with full explanation) $\lim_{n\to\infty} p_{12}^{(n)}$, or prove this limit does not exist.

5. **[5]** Let $\{p_{ij}\}$ be the transition probabilities for an irreducible Markov chain with state space S. Let $i, j, k, \ell \in S$. Suppose $\lim_{n\to\infty} p_{k\ell}^{(n)} = 0$. Prove that $\lim_{n\to\infty} p_{ij}^{(n)} = 0$.
[Hint: since $k \to i$ and $j \to \ell$, there are times $r, s \in \mathbf{N}$ with $p_{ki}^{(r)} > 0$ and $p_{j\ell}^{(s)} > 0$.]

[END OF EXAMINATION: total points = 45]