STA447/2006 (Stochastic Processes), Winter 2018

Homework #3

(10 questions; 2 pages; total points = 75)

Due: In class by 6:15 p.m. **sharp** on Thursday March 29. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. Let $\{X_n\}$ be the usual Gambler's ruin Markov chain with $p = 1/2$, $X_0 = a = 5$, and $c = 20$. Let $T = \inf\{n \geq 0 : X_n = 0 \text{ or } c\}$ as usual, and let $U = T - 1$.

(a) [**2**] Compute $\mathbf{E}(X_T)$.

(b) [**5**] Compute $\mathbf{E}(X_U)$. [Hint: If $X_T = c$, then what must X_U equal? Or, if $X_T = 0$, then what must X_U equal?]

(c) [**3**] Determine if $\mathbf{E}(X_T) = \mathbf{E}(X_0)$, and if $\mathbf{E}(X_U) = \mathbf{E}(X_0)$. Relate these facts to the Optional Stopping Corollary.

2. [**5**] Let $\{X_n\}$ be simple symmetric random walk, with $X_0 = 0$, and let $S = \inf\{n \geq 0 : X_n = -5\}$. Prove that $\lim_{M\to\infty} \mathbf{E}(X_M \mid S > M) = \infty$. [Hint: Perhaps let $T_M = \min(S, M)$.]

3. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, \ldots, 100\}$, with initial probabilities given by $\nu_{30} = \nu_{40} = 1/2$, and with transition probabilities given by $p_{1,1} = p_{100,100} = 1$, $p_{99,100} = p_{99,98} = 1/2$, and for $2 \leq i \leq 98$, $p_{i,i-1} = 2/3$ and $p_{i,i+2} = 1/3$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } 100\}$.

(a) [**3**] Compute $\mathbf{P}(X_2 = 41)$.

(b) [**4**] Determine whether or not $\{X_n\}$ is a martingale.

(c) [**4**] Compute $\mathbf{P}(X_T = 1)$. [Hint: Don't forget the Optional Stopping Corollary.]

4. Suppose we repeatedly flip a fair coin.

(a) [**5**] Compute the expected value of the number of flips until we first see the pattern "HTHT", (i) using mean recurrence times, and (ii) using martingales.

(b) [**5**] Repeat part (a) for the pattern "HTTH".

5. Consider a branching process with $X_0 = 2$, and with offspring distribution μ given by $\mu\{0\} = 1/2$ and $\mu\{1\} = 1/3$ and $\mu\{2\} = 1/6$.

(a) [**4**] Compute $\mathbf{P}(X_1 = i)$ for all $i \in \{0, 1, 2, \ldots\}$.

(b) [**4**] Compute $\mathbf{P}(X_2 = 1)$.

6. [**4**] Suppose a stock costs \$20 today, and tomorrow will cost either \$50 with probability 2/3, or \$10 with probability 1/3. Compute (with explanation) the fair (no-arbitrage) price of a European call option to buy the stock tomorrow for \$30, **(a)** using Profit Computation, and also **(b)** using the Martingale Pricing Principle.

7. [**4**] Let $\{B_t\}_{t\geq 0}$ be Brownian motion. Compute $\mathbf{Var}(B_5 B_8)$. [Hint: You may use without proof that if $Z \sim \text{Normal}(0, 1)$, then $\mathbf{E}(Z) = \mathbf{E}(Z^3) = 0$, $\mathbf{E}(Z^2) = 1$, and $\mathbf{E}(Z^4) = 3$.]

8. [**4**] Let $\{B_t\}_{t\geq 0}$ be Brownian motion. Let $\theta \in \mathbf{R}$, and let $Z_t = \exp(\theta B_t - \theta^2 t/2)$. Prove that $\{Z_t\}_{t\geq 0}$ is a martingale. [Hint: You may use without proof that if $Z \sim \text{Normal}(0, 1)$, and $a \in \mathbf{R}$, then $\mathbf{E}[e^{aZ}] = e^{a^2/2}$.]

9. Let $\{B_t\}_{t\geq 0}$ be Brownian motion, let $X_t = x_0 \exp(\mu t + \sigma B_t)$ be the stock price model (where $\sigma > 0$), and let $D_t = e^{-rt} X_t$ be the discounted stock price.

(a) [**2**] Show that if $\mu = r - \frac{\sigma^2}{2}$, then $\{D_t\}$ is a martingale. [Hint: Don't forget the previous question.]

(b) [**6**] Show that if $\mu = r - \frac{\sigma^2}{2}$, then

$$\mathbf{E}\Big[e^{-rS} \max(0,\, X_S - K)\Big] \;=\; X_0\, \Phi\left(\frac{(r + \frac{\sigma^2}{2})S - \log(K/X_0)}{\sigma\sqrt{S}}\right) - e^{-rS} K\, \Phi\left(\frac{(r - \frac{\sigma^2}{2})S - \log(K/X_0)}{\sigma\sqrt{S}}\right),$$

where $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\, dv$ is the cdf of a standard normal distribution. [Hint: Write the expectation as an integral with respect to the density function for B_S. Then, break up the integral into the part where $X_S - K \geq 0$ and the part where $X_S - K < 0$.]

[Note: This is the famous "Black-Scholes formula". You do <u>not</u> need to memorise it.]

10. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda = 3$.

(a) [**3**] Compute $\mathbf{P}[N(6) = 2 \,|\, N(8) = 4]$.

(b) [**4**] Compute $\mathbf{P}[N(6) = 2 \,|\, N(8) = 4,\; N(3) = 1]$.

(c) [**4**] Compute $\mathbf{E}\Big([N(8) - N(5)]\,[N(7) - N(2)]\Big)$. [Hint: You may use without proof the fact that if $Y \sim \text{Poisson}(m)$, then $\mathbf{E}(Y) = \mathbf{Var}(Y) = m$.]

[END; total points = 75]