<u>**STA447/2006 (Stochastic Processes), Winter 2018**</u>

<u>**Homework #2**</u>

(5 questions; 2 pages; total points = 50)

Due: In class by 6:15 p.m. <u>**sharp**</u> on Thursday March 8. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

Include at the top of the first page: Your <u>name</u> and <u>student number</u>.

1. Consider a (discrete-time) Markov chain $\{X_n\}$ on the state space $S = \{1, 2, 3, 4\}$, with transition probabilities given by

$$(p_{ij}) = \begin{pmatrix} 0 & 0 & 2/3 & 1/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) [**2**] Draw a picture of this Markov chain.

(b) [**2**] Is this chain irreducible?

(c) [**2**] Is this chain periodic? If yes, with what period?

(d) [**4**] Find a stationary distribution $\{\pi_i\}$ for this Markov chain.

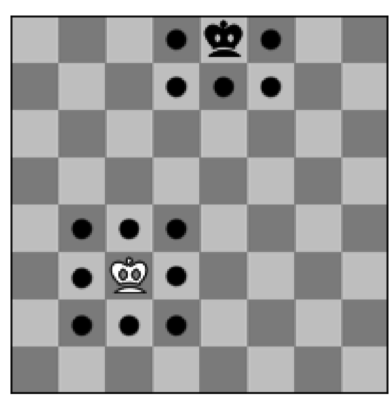
(e) [**2**] Is this chain reversible with respect to that $\{\pi_i\}$?

(f) [**3**] Determine whether each of the following statements is true or false:

(i) $\lim_{n\to\infty} p_{12}^{(n)} = \pi_2$.

(ii) $\lim_{n\to\infty} \frac{1}{2}[p_{12}^{(n)} + p_{12}^{(n+1)}] = \pi_2$.

(iii) $\lim_{n\to\infty} \frac{1}{n}[p_{12}^{(1)} + p_{12}^{(2)} + \ldots + p_{12}^{(n)}] = \pi_2$.



2. [**8**] In chess, a "king" can move one square in any direction (horizontal, vertical, or diagonal) as long as it does not leave the 8×8 board. (For example, in the diagram to the right, the white king can move to 8 different squares, while the black king can only move to 5 different squares.) Consider a king chess piece all alone on a usual 8×8 chessboard. Suppose the king begins in the lower-left corner, and then repeatedly moves to one of its adjacent squares (including diagonals) with equal probability. Let X_n be its position after n moves. Compute (with full explanation) $\lim_{n\to\infty} \mathbf{P}(X_n = i)$ for each of the 64 squares i. [Hint: Is this a random walk on a graph?] Optional bonus (not graded): what about a knight?

3. [4] Let $s(a) = \mathbf{P}_a(T_c < T_0)$ and $r(a) = \mathbf{P}_a(T_0 < T_c)$ be the probabilities of winning and losing at Gambler's Ruin, respectively. Using the formulas derived in class, verify with explicit algebra that $r(a) + s(a) = 1$, for any $0 < p < 1$ and $0 < a < c$. What can you conclude from this?

4. [A special case of the *Gibbs Sampler*.] Let $S = \mathbf{Z} \times \mathbf{Z}$, and let $f : S \to (0, \infty)$ be some function from S to the positive real numbers. Let $K = \sum_{(x,y)\in S} f(x, y)$, and assume that $K < \infty$. For $x, y \in \mathbf{Z}$, let $C(x) = \sum_{w\in\mathbf{Z}} f(x, w)$, and $R(y) = \sum_{z\in\mathbf{Z}} f(z, y)$. Consider an algorithm which proceeds at each time n as follows. Given a pair $(X_{n-1}, Y_{n-1}) \in S$ at time $n - 1$, it chooses either the "horizontal" or "vertical" option, with probability 1/2 each. If it chooses horizontal, then it sets $Y_n = Y_{n-1}$, and chooses X_n randomly to equal x with probability $f(x, Y_{n-1}) / R(Y_{n-1})$ for each $x \in \mathbf{Z}$. If it chooses vertical, then it sets $X_n = X_{n-1}$, and chooses Y_n randomly to equal y with probability $f(X_{n-1}, y) / C(X_{n-1})$ for each $y \in \mathbf{Z}$.

(a) [4] Verify that this algorithm produces a sequence $\{(X_n, Y_n)\}_{n\in\mathbf{N}}$ of random pairs which is a Markov chain on S, with transition probabilities given by

$$p_{(x,y),(z,w)} \;=\; \begin{cases} \frac{f(z,w)}{2\,C(x)} + \frac{f(z,w)}{2\,R(y)}\,, & x = z \text{ and } y = w \\ \frac{f(z,w)}{2\,C(x)}\,, & x = z \text{ and } y \neq w \\ \frac{f(z,w)}{2\,R(y)}\,, & x \neq z \text{ and } y = w \\ 0\,, & \text{otherwise} \end{cases}$$

(b) [3] Verify directly that $\sum_{(z,w)\in S} p_{(x,y),(z,w)} = 1$ for all $(x, y) \in S$.

(c) [3] Show that the chain is reversible with respect to $\pi_{(x,y)} = \frac{f(x,y)}{K}$.

(d) [5] Compute $\lim_{n\to\infty} p^{(n)}_{(x,y),(z,w)}$ for all $x, y, z, w \in \mathbf{Z}$ (carefully justifying each step).

5. Let $\{Z_i\}_{i=1}^{\infty}$ be an i.i.d. collection of random variables with $\mathbf{P}[Z_i = -1] = 3/4$ and $\mathbf{P}[Z_i = C] = 1/4$, for some $C > 0$. Let $X_0 = 5$, and $X_n = 5 + Z_1 + Z_2 + \ldots + Z_n$ for $n \geq 1$. Finally, let $T = \inf\{n \geq 1 : X_n = 0 \text{ or } Z_n > 0\}$.

(a) [3] Find (with explanation) a value of C such that $\{X_n\}$ is a martingale.

(b) [2] For this value of C, compute (with explanation) $\mathbf{E}(X_9)$.

(c) [3] For this value of C, compute (with explanation) $\mathbf{E}(X_T)$. [Hint: is T bounded?]

[END; total points = 50]