STA447/2006 Midterm, February 16, 2017

(2 hours; 5 questions; 3 pages; total points = 52)

[SOLUTIONS]

1. Suppose there are 10 lily pads arranged in a circle, numbered consecutively clockwise from 1 to 10. A frog begins on lily pad #1. Each second, the frog jumps one pad clockwise with probability 1/4, or two pads clockwise with probability 3/4.

(a) [**3**] Specify a state space S, initial probabilities $\{\nu_i\}$, and transition probabilities $\{p_{ij}\}$, with respect to which this process is a Markov chain.

Solution. *Here* $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, *and* $\nu_1 = 1$ *(with* $\nu_i = 0$ *for all other* i*). Also, for* $1 \leq i \leq 9$, $p_{i,i+1} = 1/4$, *and for* $1 \leq i \leq 8$, $p_{i,i+2} = 3/4$, *and* $p_{10,1} = 1/4$, *and* $p_{9,1} = p_{10,2} = 3/4$, *with* $p_{ij} = 0$ *otherwise.*

(b) [**2**] Determine if this Markov chain is irreducible.

Solution. *Yes, since it is always possible to move one space clockwise, and hence eventually get to every other state with positive probability.*

(c) [**2**] Determine if this Markov chain is aperiodic, or if not then what its period equals.

Solution. *From any state* i*, it is possible to return in 10 seconds by moving one pad clockwise at each jump, or to return in 9 seconds by moving two pads clockwise on the first jump and then one pad clockwise for 8 additional jumps. Since* $gcd(10, 9) = 1$*, the chain is aperiodic.*

(d) [**3**] Determine whether or not $\sum_{n=1}^{\infty} p_{15}^{(n)} = \infty$.

Solution. *Since the chain is irreducible, and the state space is finite, by the Finite Space Theorem we have* $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ *for all* $i, j \in S$*, so in particular* $\sum_{n=1}^{\infty} p_{15}^{(n)} = \infty$.

(e) [**3**] Either find a stationarity distribution $\{\pi_i\}$ for this chain, or prove that no stationary distribution exists.

Solution. *For every state* $j \in S$, $\sum_{i \in S} p_{ij} = (1/4) + (3/4) = 1$*. Hence, the chain is* <u>*doubly stochastic*</u>*. So, since* $|S| < \infty$*, the* <u>*uniform*</u> *distribution on* S *is a stationary distribution. Hence, we can take* $\pi_1 = \pi_2 = \ldots = \pi_{10} = 1/10$.

(f) [**2**] Determine whether or not $\lim_{n\to\infty} p_{15}^{(n)}$ exists, and if so what it equals.

Solution. *Yes, since the chain is irreducible and aperiodic with stationary distribution* $\{\pi_i\}$*, therefore* $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j = 1/10$ *for all* $i, j \in S$*, and in particular* $\lim_{n\to\infty} p_{15}^{(n)} = \pi_5 = 1/10$.

(g) [**2**] Determine whether or not $\lim_{n\to\infty} \frac{1}{2}[p_{15}^{(n)} + p_{15}^{(n+1)}]$ exists, and if so what it equals.

Solution. *Since* $\lim_{n\to\infty} p_{15}^{(n)} = 1/10$*, therefore also* $\lim_{n\to\infty} p_{15}^{(n+1)} = 1/10$*, and hence also* $\lim_{n\to\infty} \frac{1}{2}[p_{15}^{(n)} + p_{15}^{(n+1)}] = 1/10$.

2. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j\in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) [**3**] There is a state $k \in S$ such that if the chain is started at k, then there is a positive probability that the chain will visit k exactly twice more (and then never again).

Solution. *Yes. For example, let* $S = \{1, 2\}$*, with* $p_{11} = 1/3$*,* $p_{12} = 2/3$*, and* $p_{22} = 1$ *(with* $p_{ij} = 0$ *otherwise). Then if the chain is started at* $k = 1$*, then it will initially follow the path* $1 \to 1 \to 1 \to 2 \to 2 \to 2 \to \ldots$ *with probability* $(1/3)(1/3)(2/3)(1)(1)\ldots >= 2/27 > 0$.

(b) [**3**] The chain is irreducible and transient, and there are $k, \ell \in S$ with $f_{k\ell} = 1$.

Solution. *Yes. For example, consider simple random walk with* $p = 3/4$*, so* $S = \mathbf{Z}$ *and* $p_{i,i+1} = 3/4$ *and* $p_{i,i-1} = 1/4$ *for all* $i \in S$ *(with* $p_{ij} = 0$ *otherwise). Let* $k = 0$ *and* $\ell = 5$*. Then as shown in class,* $f_{05} = 1$*, and the chain is irreducible and transient. (Of course,* S *is infinite here; if* S *is finite then all irreducible chains are recurrent.)*

(c) [**3**] The chain is irreducible and transient, and is reversible with respect to some probability distribution π.

Solution. *Does not exist. Indeed, if the chain is reversible with respect to* π*, then* π *is a stationarity distribution. Then if it is also irreducible, then by the Stationarity Recurrence Lemma, it is recurrent, i.e. it is not transient.*

3. [**6**] Let $S = \{1, 2, 3, \ldots\}$, with $\pi_i = 2/3^i$ for all $i \in S$. Find (with proof) explicit transition probabilities $\{p_{ij}\}_{i,j\in S}$ such that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$. [Hint: Don't forget the Metropolis (MCMC) algorithm.]

Solution. *The Metropolis algorithm says that for* $j = i\pm1$ *we want* $p_{ij} = (1/2)\min(1, \pi_j/\pi_i)$*. So, for all* $i \geq 1$*, we set* $p_{i+1,i} = 1/2$*, and* $p_{i,i+1} = (1/2)[(2/3^{i+1})/(2/3^i)] = (1/2)[1/3] = 1/6$*. Then* $p_{11} = 1-p_{12} = 1-(1/6) = 5/6$*, and for* $i \geq 2$*,* $p_{ii} = 1-p_{i,i-1}-p_{i,i+1} = 1-(1/2)-(1/6) = 1/3$*, with* $p_{ij} = 0$ *otherwise. Then by construction,* $\pi_i p_{ij} = \pi_j p_{ji}$ *for all* $i, j \in S$*, the chain is reversible with respect to* π*, and hence* π *is a stationary distribution. Also, the chain is irreducible since it is always possible to increase or decrease by 1. And, the chain is aperiodic since e.g.* $p_{11} > 0$*. Hence,* $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ *for all* $i, j \in S$.

4. [**5**] Consider the undirected graph with vertex set $V = \{1, 2, 3, 4, 5\}$, and an undirected edge (of weight 1) between each of the following six pairs of vertices (and no other edges): (1,2), (2,3), (3,4), (4,5), (1,3), and (3,5). Let $\{p_{ij}\}_{i,j\in V}$ be the transition probabilities for random walk on this graph. Compute (with full explanation) $\lim_{n\to\infty} p_{23}^{(n)}$, or prove that this limit does not exist.

Solution. *The graph is connected (since we can get from $1 \to 2 \to 3 \to 4 \to 5$ and back), so the walk is irreducible. Also, the walk is aperiodic since e.g. we can get from 1 to 1 in 2 steps by $1 \to 2 \to 1$, or in 3 steps by $1 \to 3 \to 2 \to 1$, and $gcd(2,3) = 1$. Here $Z = \sum_u d(u) = 2|E| = 2(6) = 12 < \infty$. Hence, as shown in class, if $\pi_u = d(u)/Z = d(u)/12$, then the walk is reversible with respect to π, so π is a stationary distribution. Also $d(3) = 4$, because there are four edges from the vertex 3. Hence, by the Graph Convergence Theorem, $\lim_{n\to\infty} p_{23}^{(n)} = \pi_3 = d(3)/12 = 4/12 = 1/3$.*

5. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, \ldots\}$ of all positive integers, which is also a <u>martingale</u>. Assume $X_0 = 5$, and that there is $c > 0$ such that $p_{i,i-1} = c$ and $p_{i,i+2} = 1 - c$ for all $i \geq 2$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } X_n \geq 10\}$.

(a) [**3**] Determine (with explanation) what c must equal. [Hint: remember that $\{X_n\}$ is a martingale.]

Solution. *For $i \geq 2$, we need $\sum_j j p_{ij} = i$, so $(i-1)c + (i+2)(1-c) = i$, so $i + 2 - 3c = i$, so $2 = 3c$, so $c = 2/3$.*

(b) [**3**] Determine (with explanation) what p_{11} must equal. [Hint: again, remember that $\{X_n\}$ is a martingale.]

Solution. *We need $\sum_{j\in S} j p_{1j} = 1$. But $\sum_{j\in S} j p_{1j} = \sum_{j=1}^{\infty} j p_{1j} = p_{11} + \sum_{j=2}^{\infty} j p_{ij} \geq p_{11} + \sum_{j=2}^{\infty} 2 p_{ij} = p_{11} + 2(1 - p_{11}) = 2 - p_{11}$. For this to equal 1, we need $p_{11} = 1$.*

(c) [**3**] Determine (with explanation) the value of $\mathbf{E}(X_3)$.

Solution. *Since $\{X_n\}$ is a martingale, $\mathbf{E}(X_n) = \mathbf{E}(X_0) = 5$ for all n, so in particular $\mathbf{E}(X_3) = 5$.*

(d) [**3**] Determine (with explanation) the value of $\mathbf{E}(X_T)$.

Solution. *Clearly the chain is bounded up to time T, indeed we always have $|X_n| \mathbf{1}_{n\leq T} \leq 11$. Hence, by the Optional Stopping Corollary, $\mathbf{E}(X_T) = \mathbf{E}(X_0) = 5$.*

(e) [**3**] Prove or disprove that $\sum_{n=1}^{\infty} p_{55}^{(n)} = \infty$.

Solution. *Starting at 5, the chain has positive probability of immediately going $5 \to 4 \to 3 \to 2 \to 1$ and then getting stuck at 1 forever and never returning to 5. Hence, the state 5 is <u>transient</u>. It thus follows from the Recurrence Theorem that $\sum_{n=1}^{\infty} p_{55}^{(n)} < \infty$, i.e. $\sum_{n=1}^{\infty} p_{55}^{(n)} \neq \infty$. (Note that this chain is <u>not</u> irreducible, since e.g. $f_{12} = 0$, so the Cases Theorem etc do <u>not</u> apply.)*

[END OF EXAMINATION: total points = 52]