STA447/2006 (Stochastic Processes), Winter 2017

Homework #2

(10 questions; 4 pages; total points = 88)

Due: In class by 6:10 p.m. **sharp** on Thursday March 16. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. Consider a (discrete-time) Markov chain $\{X_n\}$ on the state space $S = \{1,2,3,4\}$, with transition probabilities given by

$$(\,p_{ij}\,) \;=\; \begin{pmatrix} 0 & 0 & 1/4 & 3/4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) [2] Draw a picture of this Markov chain.

(b) [2] Is this chain irreducible?

(c) [2] Is this chain periodic? If yes, with what period?

(d) [4] Find a stationary distribution $\{\pi_i\}$ for this Markov chain.

(e) [2] Is this chain reversible with respect to that $\{\pi_i\}$?

(f) [3] Determine whether each of the following statements is true or false:

(i) $\lim_{n\to\infty} p_{12}^{(n)} = \pi_2$.

(ii) $\lim_{n\to\infty} \frac{1}{2}[p_{12}^{(n)} + p_{12}^{(n+1)}] = \pi_2$.

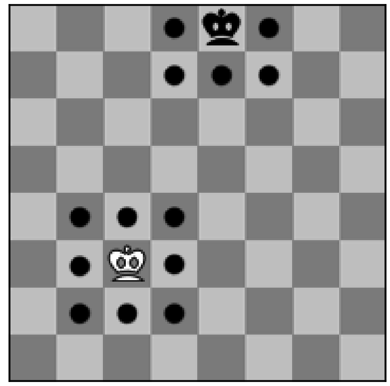
(iii) $\lim_{n\to\infty} \frac{1}{n}[p_{12}^{(1)} + p_{12}^{(2)} + \ldots + p_{12}^{(n)}] = \pi_2$.

2. Consider the undirected graph on the vertices $V = \{1,2,3,4,5\}$, with weights given by $w(1,2) = w(2,1) = w(2,3) = w(3,2) = w(1,3) = w(3,1) = w(3,4) = w(4,3) = w(3,5) = w(5,3) = 1$, and $w(u,v) = 0$ otherwise.

(a) [2] Draw a picture of this graph.

(b) [4] Compute (with full explanation) $\lim_{n\to\infty} \mathbf{P}[X_n = 3]$, where $\{X_n\}$ is the usual (simple) random walk on this graph.

3. **[6]** In chess, a "king" can move one square in any direction (horizontal, vertical, or diagonal) as long as it does not leave the 8×8 board. (For example, in the diagram to the right, the white king can move to 8 different squares, while the black king can only move to 5 different squares.) Consider a king chess piece all alone on a usual 8×8 chessboard. Suppose the king begins in the lower-left corner, and then each second moves to one of its adjacent squares (including diagonals) with equal probability. Compute (with explanation) $\lim_{n\to\infty} \mathbf{P}(X_n = i)$ for each of the 64 squares i. [Hint: Is this a random walk on a graph?]



4. **[4]** Let $s(a) = \mathbf{P}_a(T_c < T_0)$ and $r(a) = \mathbf{P}_a(T_0 < T_c)$ be the probabilities of winning and losing at Gambler's Ruin, respectively. Verify by explicit algebra that $r(a) + s(a) = 1$, for any $0 < p < 1$ and $0 < a < c$. What can you conclude from this?

5. [A special case of the *Gibbs Sampler*.] Let $S = \mathbf{Z} \times \mathbf{Z}$, and let $f : S \to (0, \infty)$ be some function from S to the positive real numbers. Let $K = \sum_{(x,y)\in S} f(x, y)$, and assume that $K < \infty$. For $x, y \in \mathbf{Z}$, let $C(x) = \sum_{w\in\mathbf{Z}} f(x, w)$, and $R(y) = \sum_{z\in\mathbf{Z}} f(z, y)$. Consider an algorithm which proceeds at each time n as follows. Given a pair $(X_{n-1}, Y_{n-1}) \in S$ at time $n-1$, it chooses either the "horizontal" or "vertical" option, with probability 1/2 each. If it chooses horizontal, then it sets $Y_n = Y_{n-1}$, and chooses X_n randomly to equal x with probability $f(x, Y_{n-1}) / R(Y_{n-1})$ for each $x \in \mathbf{Z}$. If it chooses vertical, then it sets $X_n = X_{n-1}$, and chooses Y_n randomly to equal y with probability $f(X_{n-1}, y) / C(X_{n-1})$ for each $y \in \mathbf{Z}$.

(a) **[4]** Verify that this algorithm produces a sequence $\{(X_n, Y_n)\}_{n\in\mathbf{N}}$ of random pairs which is a Markov chain on S, with transition probabilities given by

$$p_{(x,y),(z,w)} = \begin{cases} \frac{f(z,w)}{2\,C(x)} + \frac{f(z,w)}{2\,R(y)}\,, & x = z \text{ and } y = w \\ \frac{f(z,w)}{2\,C(x)}\,, & x = z \text{ and } y \neq w \\ \frac{f(z,w)}{2\,R(y)}\,, & x \neq z \text{ and } y = w \\ 0\,, & \text{otherwise} \end{cases}$$

(b) **[3]** Verify directly that $\sum\limits_{(z,w)\in S} p_{(x,y),(z,w)} = 1$ for all $(x, y) \in S$.

(c) **[3]** Show that the chain is reversible with respect to $\pi_{(x,y)} = \frac{f(x,y)}{K}$.

(d) **[5]** Compute $\lim\limits_{n\to\infty} p^{(n)}_{(x,y),(z,w)}$ for all $x, y, z, w \in \mathbf{Z}$ (carefully justifying each step).

6. Let $\{Z_i\}_{i=1}^{\infty}$ be an i.i.d. collection of random variables with $\mathbf{P}[Z_i = -1] = 3/4$ and $\mathbf{P}[Z_i = C] = 1/4$, for some $C > 0$. Let $X_0 = 5$, and $X_n = 5 + Z_1 + Z_2 + \ldots + Z_n$ for $n \geq 1$. Finally, let $T = \inf\{n \geq 1 : X_n = 0 \text{ or } Z_n > 0\}$.

(a) [**3**] Find (with explanation) a value of C such that $\{X_n\}$ is a martingale.

(b) [**2**] For this value of C, compute (with explanation) $\mathbf{E}(X_9)$.

(c) [**3**] For this value of C, compute (with explanation) $\mathbf{E}(X_T)$. [Hint: is T bounded?]

7. Consider simple random walk with $X_0 = 2$ and $p = 2/3$.

(a) [**3**] Compute $\mathbf{P}(X_2 = i)$ for all $i \in \mathbf{Z}$.

(b) [**3**] Compute $\mathbf{E}(X_2)$ using the explicit formula $\sum_i i\,\mathbf{P}(X_2 = i)$, together with the probability values found in part (a).

(c) [**4**] Compute $\mathbf{E}(X_2)$ using Wald's Theorem (with explanation), and see if you get the same answer as in part (b).

8. Consider a branching process with $X_0 = 2$, and with offspring distribution μ satisfying that $\mu\{0\} = 1/2$ and $\mu\{1\} = 1/3$ and $\mu\{2\} = 1/6$.

(a) [**4**] Compute $\mathbf{P}(X_1 = i)$ for all $i \in \{0, 1, 2, \ldots\}$.

(b) [**4**] Compute $\mathbf{P}(X_2 = 1)$.

9. Complete the proof of the Cyclic Decomposition Lemma, by the following steps. Assume $\{p_{ij}\}$ are the transition probabilities for an irreducible Markov chain on a state space S with period $b \geq 2$, fix $i_0 \in S$, and let $S_r = \{j \in S : p_{i_0 j}^{(bm+r)} > 0$ for some $m \in \mathbf{N}\}$.

(a) [**3**] Show that $\bigcup_{r=0}^{b-1} S_r = S$. [Hint: use irreducibility.]

(b) [**4**] Show that if $0 \leq r < t \leq b-1$, then S_r and S_t are disjoint, i.e. $S_r \cap S_t = \emptyset$. [Hint: Suppose $j \in S_r \cap S_t$. Find $m \in \mathbf{N}$ with $p_{ji}^{(m)} > 0$. What can you conclude about $\gcd\{n \geq 1 : p_{ii}^{(n)} > 0\}$?]

(c) [**2**] Suppose $i \in S_r$ for some $0 \leq r \leq b-2$, and $p_{ij} > 0$. Prove that $j \in S_{r+1}$.

(d) [**1**] Suppose $i \in S_{b-1}$, and $p_{ij} > 0$. Prove that $j \in S_0$.

(e) [**3**] Let $\hat{P} = P^{(b)}$, i.e. $\hat{p}_{ij} = p_{ij}^{(b)}$, corresponding to b steps of the original chain, except restricted to just S_0. Prove that $\hat{P}$ is irreducible. [Hint: Suppose there are $i, j \in S_0$ with $\hat{p}_{ij}^{(n)} = 0$ for all $n \geq 1$. What does this say about f_{ij} for the original chain?]

(f) [**3**] Prove that $\hat{P}$ is aperiodic. [Hint: Suppose there is $i \in S_0$ with $gcd\{n \geq 1 : \hat{p}_{ii}^{(n)} > 0\} = m \geq 2$. What does this say about the period of i in the original chain?]

10. [BONUS QUESTION. WORTH UP TO 10 POINTS, BUT ONLY FOR A COMPLETE SOLUTION. PLEASE DO NOT HAND THIS QUESTION IN UNLESS YOU HAVE A COMPLETE SOLUTION.] Complete the proof of the Periodic Convergence Theorem, by the following steps. Assume $\{p_{ij}\}$ are the transition probabilities for an irreducible Markov chain on a state space S with period $b \geq 2$, and stationary distribution $\{\pi_i\}$. Let $S_0, S_1, \ldots, S_{b-1}$ be as in Cyclic Decomposition Lemma, with $\hat{P}$ the Markov chain corresponding to $P^{(b)}$ restricted to S_0. For any subset $A \subseteq S$, let $\pi(A) = \sum_{i \in A} \pi_i$ be the total probability of A according to π.

(a) Prove that $\pi(S_0) = \pi(S_1) = \ldots = \pi(S_{b-1}) = 1/b$. [Hint: What is the relationship in this case between $\mathbf{P}[X_0 \in S_0]$ and $\mathbf{P}[X_1 \in S_1]$? On the other hand, if we begin in stationarity, then how does $\mathbf{P}[X_n \in S_0]$ change with n?]

(b) Let $\hat{\pi}_i = b\,\pi_i$ for all $i \in S_0$. Show that $\hat{\pi}$ is a stationary distribution for $\hat{P}$.

(c) Conclude that $\lim_{m\to\infty} \hat{p}_{ij}^{(m)} = \hat{\pi}_j$ for all $i, j \in S_0$, i.e. $\lim_{m\to\infty} p_{ij}^{(bm)} = b\,\pi_j$.

(d) Argue that similarly $\lim_{m\to\infty} p_{ij}^{(bm)} = b\,\pi_j$ for all $i, j \in S_r$, for any $1 \le r \le b-1$.

(e) Show that for $i \in S_0$ and $j \in S_r$ for any $1 \le r \le b-1$, we have $\lim_{m\to\infty} p_{ij}^{(bm+r)} = b\,\pi_j$. [Hint: $p_{ij}^{(bm+r)} = \sum_{k\in S} p_{ik}^{(r)} p_{kj}^{(bm)}$.]

(f) Conclude that for $i \in S_0$, $\lim_{m\to\infty} \frac{1}{b}[p_{ij}^{(bm)} + p_{ij}^{(bm+1)} + \ldots + p_{ij}^{(bm+b-1)}] = \pi_j$ for any $j \in S$.

(g) Show that the previous statement holds for any $i \in S$ (not just $i \in S_0$).

(h) Show that $\lim_{n\to\infty} \frac{1}{b}\,[p_{ij}^{(n)} + \ldots + p_{ij}^{(n+b-1)}] = \pi_j$, i.e. that we can take the limit over all n not just $n = bm$. [Hint: for any n, let $m = \lfloor n/b \rfloor$.]

[END; total points = 88]