Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**____________________

Class (circle one): STA447 STA2006

UNIVERSITY OF TORONTO
Faculty of Arts and Science

STA447/2006H1 (Stochastic Processes)
MIDTERM TEST
February 25, 2016, 6:10 p.m.

Duration: 120 minutes. Total points: 60.

Aids allowed: NONE.

This examination paper consists of **7** single-sided pages (including this cover page), and **5** questions. The backs of the pages can be used to continue an answer (be sure to INDICATE THIS), or as scrap paper. The value of each question is indicated in [square-brackets].

You may use results from class lectures, with explanation.

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO DO SO.

For graders' use only:

	Score
1 (10)	
2 (27)	
3 (6)	
4 (6)	
5 (11)	
Total (60)	

1. Consider a Markov chain on the state space $S = \{1, 2, 3, 4\}$ with the following transition matrix:

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.2 & 0.3 \end{pmatrix}$$

Let π be the uniform distribution on S, so $\pi_i = 1/4$ for all $i \in S$.

(a) [2] Compute $p_{14}^{(2)}$.

(b) [2] Is this Markov chain reversible with respect to π?

(c) [3] Is π a stationary distribution for this Markov chain?

(d) [3] Does $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$? Why or why not?

2. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

   (a) [3] The chain is irreducible and periodic (i.e., not aperiodic), and has a stationary probability distribution.

   (b) [3] The chain is irreducible, and there are states $k \in S$ having period 2, and $\ell \in S$ having period 4.

   (c) [3] There are distinct states $k, \ell \in S$ such that if the chain is started at k, then there is a positive probability that the chain will visit ℓ exactly five times (and then never again).

(d) [3] The chain is irreducible and transient, and there are $k, \ell \in S$ with $f_{k\ell} = 1$.

(e) [3] The chain is irreducible and transient, and is reversible with respect to some probability distribution π.

(f) [3] The chain is irreducible and has a stationary probability distribution π, and $p_{ij} < 1$ for all $i, j \in S$, but the chain is <u>not</u> reversible with respect to π.

(g) [3] The chain is irreducible and transient, and there are $k, \ell \in S$ with $p_{k\ell}^{(n)} \geq 1/3$ for all $n \in \mathbf{N}$.

(h) [3] The chain is irreducible, and there are distinct states $i, j, k, \ell \in S$ such that $f_{ij} < 1$, and $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$.

(i) [3] There are states $i, j, k \in S$ with $p_{ij} > 0$, $p_{jk}^{(2)} > 0$, and $p_{ki}^{(3)} > 0$, and the state i is periodic (i.e., has period > 1).

3. [6] Let $S = \{1, 2, 3\}$, with $\pi_1 = 1/2$ and $\pi_2 = 1/3$ and $\pi_3 = 1/6$. Find (with proof) irreducible transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that π is a stationarity distribution. [Hint: Don't forget the Metropolis (MCMC) algorithm.]

4. [6] Consider the undirected graph with vertex set $V = \{1, 2, 3, 4\}$, and an undirected edge (of weight 1) between each of the following four pairs of edges (and no other edges): (1,2), (2,3), (3,4), and (2,4). Let $\{p_{ij}\}_{i,j \in V}$ be the transition probabilities for random walk on this graph. Compute (with full explanation) $\lim_{n\to\infty} p_{21}^{(n)}$, or prove that this limit does not exist.

5. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4\}$, with $X_0 = 2$, and with transition probabilities satisfying that $p_{11} = p_{44} = 1$, $p_{21} = 1/4$, $p_{34} = 1/5$, and $p_{23} = p_{31} = p_{12} = p_{13} = p_{14} = p_{41} = p_{42} = p_{43} = 0$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } 4\}$.

   (a) [5] Find (with explanation) non-negative values of p_{22}, p_{24}, p_{32}, and p_{33}, such that $\sum_{j \in S} p_{ij} = 1$ for all $i \in S$ (as it must), and also $\{X_n\}$ is a <u>martingale</u>.

   (b) [3] For the values found in part (a), compute with justification $\mathbf{E}(X_T)$.

   (c) [3] For the values found in part (a), compute with justification $\mathbf{P}(X_T = 1)$.

End of examination
Total pages: 7
Total points: 60