Random Race Starter Timer to Reduce Anticipation

by

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1. Introduction.

Athletics competitions have the concern that a runner could “anticipate” the starter gun to gain a slight advantage. To prevent this, they judge any runner starting within 0.1 seconds of the starter gun to have committed a false start and thus be disqualified. This runs the risk of unfairly penalising a competitor for simply having a very fast reaction time.

It has been proposed[2] to replace this rule by a computer-generated randomised start time, to avoid anticipation[3]. In this paper, we investigate how random start times can reduce the anticipation advantage.

2. Set-Up.

Suppose that, under a fair and equal start, a given runner has probability p of winning (or otherwise “succeeding”) in a race. However, they might choose to start at some particular time t even before hearing the starter gun. If the gun then fires within their reaction time δ, i.e. between times t and $t+\delta$, then their win probability is multiplied by some factor $r > 1$, i.e. becomes rp. On the other hand, if the gun does not fire by time $t+\delta$, then they commit a false start and are disqualified (so their win probability becomes 0).

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[2]See the article at probability.ca/starart, and pages 190–1 of the book at probability.ca/sbl.

[3]In addition, the computer sound could be played through multiple speakers near all the running lanes, thus avoiding delays due to sound taking about 0.01 seconds to travel each 3.4 meters or 2.8 lanes.

Let $f(x)$ be the probability density function for the start time X (in seconds, perhaps after a fixed get-set period of e.g. one second). Suppose for some $t \geq 0$, the given runner has decided to use the strategy of starting when they hear the gun or at time t, whichever comes first. And suppose all other runners simply start upon hearing the gun. Then the given runner's success probability is given by

$$W \;=\; p\,\mathbf{P}[X \leq t] + r\,p\,\mathbf{P}[t < X \leq t{+}\delta] + 0\,\mathbf{P}[X > t{+}\delta] \;=\; p\int_0^t f(x)\,dx + r\,p\int_t^{t+\delta} f(x)\,dx\,.$$

This can also be written as

$$W \;=\; pF(t) + r\,p[F(t+\delta) - F(t)] \;=\; p[1+A]$$

where $F(x) = \mathbf{P}(X \leq x)$, and

$$A \;=\; A(t) \;=\; F(t) - 1 + r[F(t+\delta) - F(t)] \;=\; (r-1)[(1-F(t)] - r[1 - F(t+\delta)]$$

is the "anticipation advantage".

Or as

$$W \;=\; p[1 - G(t)] + r\,p[G(t) - G(t+\delta)] \;=\; p\Big[1 + (r-1)G(t) - rG(t+\delta)\Big] \;=\; p[1+A]$$

where $G(x) = \mathbf{P}(X > x)$, and $A = A(t) = (r-1)G(t) - rG(t+\delta)$ is the "anticipation advantage".

Our goal is to minimise $\sup_t A(t)$, the maximum possible anticipation advantage.

3. Exponential Case.

Suppose now that $X \sim \mathbf{Exp}(\lambda)$, i.e. $f(x) = \lambda e^{-\lambda x}\mathbf{1}_{x>0}$ and $F(x) = 1 - e^{-\lambda x}$. Then

$$W \;=\; p(1 - e^{-\lambda t}) + rp[(1 - e^{-\lambda(t+\delta)}) - (1 - e^{-\lambda t})] \;=\; p\Big[1 + e^{-\lambda t}\Big(-1 + r(1 - e^{-\lambda\delta}\Big)\Big]\,.$$

This anticipation advantage can be easily controlled:

Theorem. If $X \sim \mathbf{Exp}(\lambda)$, then provided that $r < 1/(\delta\lambda)$, we will always have $W < p$, i.e. there is no possible anticipation advantage.

Proof. Recall that $e^z \geq 1 + z$ for any $z \in \mathbf{R}$. So, $1 - e^z \leq -z$. It follows that

$$W \;\leq\; p\Big[1 + e^{-\lambda t}\Big(-1 + r\lambda\delta\Big)\Big] \, .$$

If $r < 1/(\delta\lambda)$, then $-1 + r\lambda\delta < 0$, so $W < p$, as claimed. ∎

For example, if the reaction time is $\delta = 0.1$ seconds, and the start time distribution has $\lambda = 1$, then provided that the win probability multiplier satisfies $r < 10$, then no anticipation advantage can be had, which is good.

4. Bounded Distributions.

The above exponential distribution is an excellent solution, except that the distribution of X is not bounded. This means that there is no limit on how long the runners might have to wait in "set" position before the race begins, which could be problematic[4].

To avoid this problem, we suppose from now on that the start times are bounded, i.e. there is $M < \infty$ with $\mathbf{P}(X \leq M) = 1$. In this case, the situation is not as good as before:

Theorem. If the start time distribution is bounded, then there is always some positive anticipation advantage.

Proof. Since the start time distribution is bounded, we must have $L := \sup\{x \geq 0 : G(x) > 0\} < \infty$. But then with $t = L - \delta$, we have $A(L - \delta) = (r - 1)G(M - \delta) > 0$, so there is a positive anticipation advantage. ∎

On the other hand, if $X \leq M$, then it follows that for $t \geq M - \delta$, $A(t) = (r - 1)\, G(t)$ is a non-increasing function of t. Hence, we can assume that $t \leq M - \delta$.

Our goal is to minimise $\sup\{A(t) : 0 \leq t \leq M - \delta\}$, by choosing appropriate bounded distributions for X on $[0, M]$. Intuitively, we want X to be more likely to be in the lower part of that interval, but still have some chance of being in the higher part too.

[4] Personal communication from Carl Georgevski, track coach at the University of Toronto.

4.1. Uniform Case (Unif).

Suppose first that $X \sim \text{Uniform}[0, M]$. Then

$$W \;=\; p(t/M) + rp(1/M)\delta \;=\; (p/M)\Big[t + r\delta\Big] \,.$$

If $t = M - \delta$, then this becomes

$$W \;=\; (p/M)\Big[M - \delta + r\delta\Big] \;=\; p\Big[1 + \delta\,\frac{r-1}{M}\Big] \,.$$

Since $r > 1$, this leads to $W > p$, i.e. to an anticipation advantage.

For example, if $\delta = 0.1$ and $r = 2$ and $M = 0.2$ (which might be approximately the case for a typical starter gun), then we get $W = p[1 + (0.1)(2-1)/0.2] = 1.5\,p$, increasing the win probability by an extra 50%.

However, this advantage decreases as M increases. For example, if $\delta = 0.1$ and $r = 2$ and $M = 3$, then we get $W = p[1 + (0.1)(2-1)/3] = 1.0333\,p$, increasing the win probability by only an extra $A = 1/30$.

4.2. Truncated Exponential Case (TEλ).

Suppose now that $X \sim \min[\mathbf{Exp}(\lambda),\, M]$, i.e. an exponential distribution except truncated at M, so $F(x) = 1 - e^{-\lambda x}$ for $0 \le x < M$ but $F(M) = 1$.

Then for $t < M - \delta$, W is the same as in the full Exponential Case above, with negative anticipation advantage over a wide range of δ and r.

However, when $t = M - \delta$, we compute that

$$W \;=\; p(1 - e^{-\lambda(M-\delta)}) + rp[1 - (1 - e^{-\lambda(M-\delta)})] \;=\; p\Big[1 + (r-1)e^{-\lambda(M-\delta)}\Big] \,.$$

In this case, we always have $W > p$, so as expected there is a (small) anticipation advantage at $t = M - \delta$. For example, if the reaction time is $\delta = 0.1$ seconds, and the start time distribution has $\lambda = 1$, with maximum $M = 3$, and anticipation factor $r = 2$, then $W = 1.05502\,p$. This is slightly worse than the uniform case.

4.3. Conditional Exponential Case (CEλ).

Suppose now that $X \sim \mathbf{Exp}(\lambda)\big|_{X \leq M}$, i.e. an exponential distribution conditional on being $\leq M$, so $F(x) = (1 - e^{-\lambda x}) \big/ (1 - e^{-\lambda M})$ for $0 \leq x \leq M$. Then

$$W \;=\; p\,\frac{1 - e^{-\lambda(M-\delta)}}{1 - e^{-\lambda M}} + rp\,\frac{1 - (1 - e^{-\lambda(M-\delta)})}{1 - e^{-\lambda M}}\,.$$

This distribution comes closest to mimicking the unbounded distribution $\mathbf{Exp}(\lambda)$ above, so it seems the most promising, as we shall see below.

5. Numerical Comparisons.

We now compare the performance of the above various bounded densities. We fix the maximum delay time at $M = 3$ seconds, and consider various reaction times $\delta = 0.1$ or 0.25, and anticipation multipliers $r = 2$ or 3. In each case, we compute $\sup\{A(t) : 0 \leq t \leq M - \delta\}$, where $A(t)$ is the anticipation advantage at time t as above. Our results are as in Figure 1.

6. Conclusion.

As can be seen from Figure 1, the Truncated Exponential with $\lambda = 2$ (TE2), and the Conditional Exponential with $\lambda = 2$ (CE2), both perform well unless both the reaction time δ and the anticipation multiplier r get large.

By contrast, the Conditional Exponential with $\lambda = 1$ (CE1) performs quite well over a wide range of values of δ and r.

We thus recommend the CE1 distribution for the Random Race Start Timer. Indeed, this is the distribution that we have implemented in our online version[5]. (The online version also adds a one-second get-set period to the beginning of the delay time, so its total delay time is $1 + X$ seconds where $X \sim$ CE1 is conditional on $M \leq 3$, i.e. its total delay time is always between 1 and 4 seconds.)

[5]Available at: probability.ca/starter

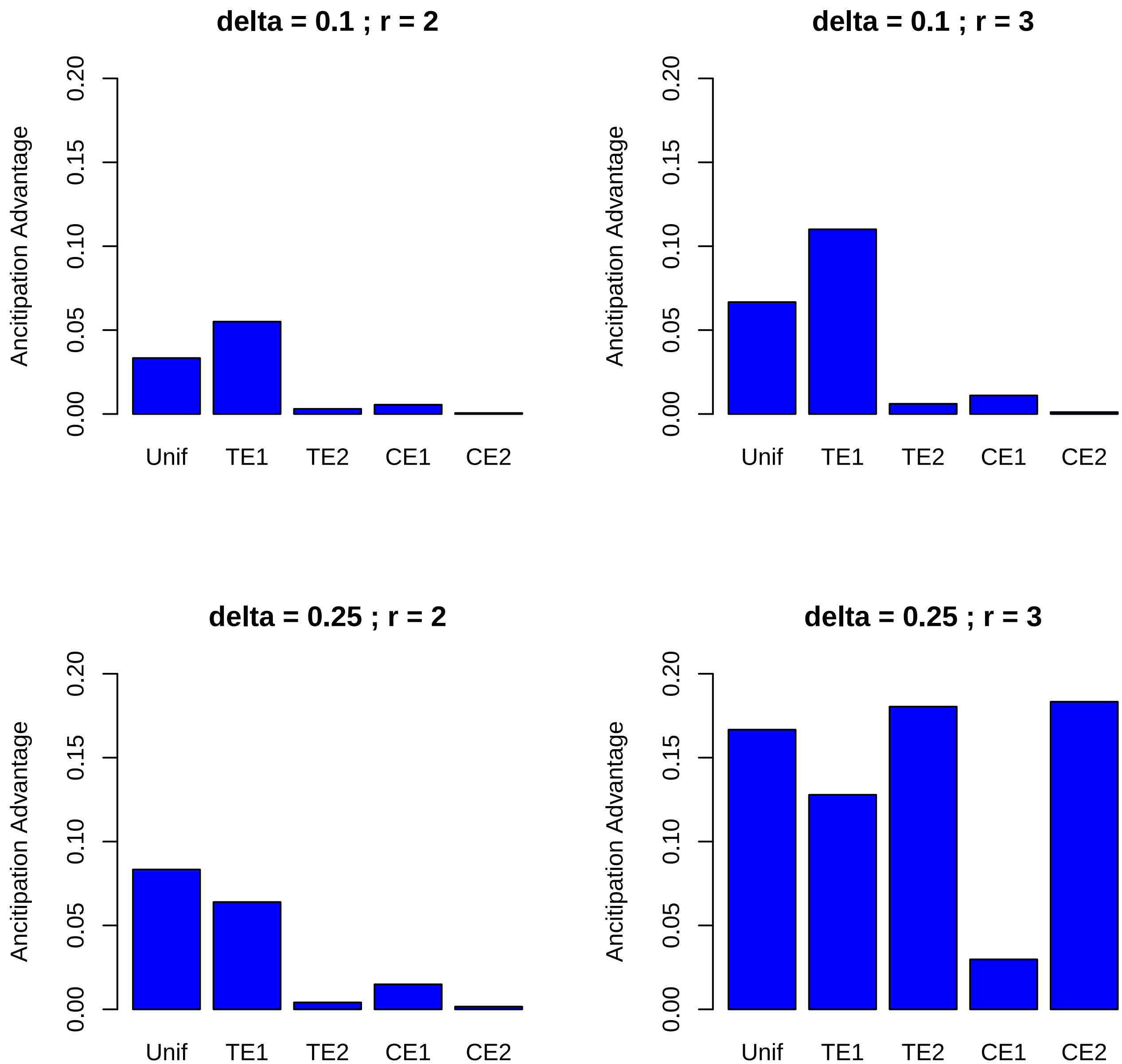

Figure 1: **Numerical comparison of the maximum anticipation advantage for various different bounded delay time distributions, with maximum delay $M = 3$, anticipation time $\delta = 0.1$ (top) or 0.25 (bottom), and anticipation multiplier $r = 2$ (left) or 3 (right), showing that CE1 is the most consistently small.**