

# Equivalence of Starting Point Cutoff and the Concentration of Hitting Times on a General State Space

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## Abstract

In this paper we extend a result of [7] to show that for a fixed discrete-time regular Markov chain on a general state space, the existence of a cutoff phenomenon for total variation distance to stationarity as the starting point tends to infinity is equivalent to the concentration of hitting times for any fixed regular set as the starting points tend to infinity. We apply this result to show that all random walks on the half-line with bounded steps exhibit starting point cutoff.

**Keywords:** Markov chains ; Cutoff phenomenon.  
**MSC2020 subject classifications:** 60.

## 1 Introduction

In the study of Markov chains, two quantities which are often of interest are hitting times of certain sets and the time it takes to converge to stationarity. In particular one is often interested in whether a Markov chain exhibits a cutoff phenomenon (see [1]): the existence of a time where the total variation distance to stationarity transitions rapidly from near one to near zero. Theorem 4.1 in [7] shows that for continuous-time Markov chains on a countable state space, when the Markov chain is started arbitrarily far from a finite set  $S$ , the existence of a cutoff phenomena is equivalent to the concentration of the hitting times of  $S$ , and in this case the hitting times are equivalent to the cutoff times. The primary goal of this paper is to establish a version of this result for discrete-time Markov chains on a general state space with the appropriate adjustments.

In Section 2 we establish our setting, notation and definitions and conclude by stating our main result (Theorem 2.6). The bulk of the proof of Theorem 2.6 is broken up into two propositions which are stated and proved in Section 4. In Section 3 we state and prove two lemmas which are the tools used to extend the proof ideas from [7] to the general state space setting. In Section 5 we use Theorem 2.6 to establish a sufficient condition for starting point cutoff which is applied in Section 6 to show that all regular random walks on the half-line with bounded steps exhibit starting point cutoff. In Section 7 we construct a Markov chain and a sequence of starting points that diverge to infinity but do not exhibit starting point cutoff, and use Theorem 2.6 to justify that it works. Finally in Section 8 we state and discuss some open problems inspired by the results of this paper. A web appendix can be found at [probability.ca/CutoffWeb](http://probability.ca/CutoffWeb) where we give a detailed proof of starting point cutoff for a toy example.

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## 2 Definitions and Statement of Equivalence

Throughout the paper we let  $P$  be a  $\phi$ -irreducible, aperiodic Markov kernel with stationary distribution  $\pi$  on a countably generated state space  $(\mathcal{X}, \mathcal{B})$ . Given  $x \in \mathcal{X}$  we can define a Markov chain  $\{X_t\}_{t \geq 0}$  starting at  $x$  with Markov kernel  $P$  by defining  $X_0 = x$  and for  $t \geq 1$  recursively setting  $X_t \sim P(X_{t-1}, \cdot)$ . We then denote the law of  $\{X_t\}_{t \geq 0}$  by  $\mathbb{P}_x$  and expectation with respect to this measure by  $\mathbb{E}_x$ . Note that for all  $t \geq 0$  and  $A \in \mathcal{B}$ ,

$$\mathbb{P}_x(X_t \in A) = P^t(x, A)$$

where  $P^t$  is the convolution of the kernel  $P$  with itself  $t$  times. For any  $S \in \mathcal{B}$  define the hitting time  $\tau_S$  of  $S$  as the random variable

$$\tau_S = \inf\{t \geq 0 : X_t \in S\}$$

Note that  $\tau_S$  is a stopping time with respect to the filtration  $\{\sigma(X_0, \dots, X_t)\}_{t \geq 0}$ . Define  $\mathcal{B}^+ := \{A \in \mathcal{B} : \pi(A) > 0\}$ . We make the additional assumption that  $P$  is regular in the sense that for any  $A \in \mathcal{B}^+$  and all  $x \in \mathcal{X}$ ,  $\mathbb{E}_x[\tau_A] < \infty$ . In fact this is not a strong additional assumption since it can be shown that a  $\phi$ -irreducible, aperiodic Markov kernel with stationary distribution  $\pi$  can be restricted to a regular kernel by throwing out a set of  $\pi$ -measure zero (see [8]).

In order to generalize Theorem 4.1 of [7] to the general state space setting we will need a candidate replacement for the role of finite sets. It turns out that for regular kernels there exists a natural replacement.

**Definition 2.1.** We call  $S \in \mathcal{B}^+$  a regular set if for any  $A \in \mathcal{B}^+$

$$\sup_{x \in S} \mathbb{E}_x[\tau_A] < \infty$$

In Chapter 11 of [8] it is shown that for  $\phi$ -irreducible regular kernels there exists regular sets. Note that in the discrete state space setting, finite sets will be regular sets since the supremum becomes a maximum (though infinite sets may or may not be regular).

Now fix a regular set  $S \in \mathcal{B}^+$ .

**Definition 2.2.** Let  $x_n$  be a sequence of starting points in  $\mathcal{X}$  and let  $t_n$  be a increasing sequence of positive reals such that  $\lim_{n \rightarrow \infty} t_n = \infty$ . We say that  $\tau_S$  concentrates at time  $t_n$  if

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n} \left( \left| \frac{\tau_S}{t_n} - 1 \right| \leq \varepsilon \right) = 1, \quad \text{for all } \varepsilon > 0$$

In other words  $\tau_S/t_n \rightarrow 1$  in probability.

The following remark provides a characterization of this phenomenon which will be convenient for our purposes.

**Remark 2.3.** It is easy to see that  $\tau_S$  concentrates at time  $t_n$  if and only if the following two conditions hold:

(i) For any  $c < 1$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < ct_n) = 0$$

(ii) For any  $c > 1$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < ct_n) = 1$$

## Starting point cutoff on a general state space

We now turn our discussion to the convergence of  $P^t(x, \cdot)$  to the stationary distribution  $\pi$ . To measure distance between these distributions we use the total variation distance.

**Definition 2.4.** For any  $x \in \mathcal{X}$  and  $t \in \mathbb{R}^+$  define

$$d_x(t) := \left\| P^{\lfloor t \rfloor}(x, \cdot) - \pi \right\|_{\text{TV}} = \sup_{A \in \mathcal{B}} \left| P^{\lfloor t \rfloor}(A, \cdot) - \pi(A) \right|$$

We say that  $d_x(t)$  is the (total variation) distance to stationarity starting at  $x$  after time  $t$ . While this makes more sense when  $t$  is a non-negative integer, defining it for any  $t \in \mathbb{R}^+$  will be convenient for defining the cutoff phenomenon. Since  $P$  is a regular aperiodic Markov kernel, it is also a aperiodic Harris recurrent Markov kernel (see Chapter 11 of [8]) and therefore  $\lim_{n \rightarrow \infty} d_x(t) = 0$  (see Chapter 13 of [8]). This is sometimes called the Harris Ergodic Theorem. While the Harris Ergodic Theorem tells you that a Markov chain converges to its stationary distribution, it gives no qualitative or quantitative information about this convergence. The cutoff phenomenon (see [1] for a review) is the observation that for many natural Markov chains (though not all) the total variation distance to stationarity stays very close to one until it hits some time (the cutoff time) where it rapidly decays to zero. There are many variations for the definition of cutoff depending on the setting and problem of interest. Often (as in the case in [1]) a sequence of Markov chains on a sequence of finite state spaces are considered and cutoff is a statement about the worst case starting points. In the general state space setting there may not be a worse case starting point, since starting points can be arbitrarily far away from the center of the distribution. Instead we consider "starting point cutoff", similar to the notion in [7] where we fix a Markov kernel and let the starting point vary.

**Definition 2.5.** Let  $x_n$  be a sequence of starting points in  $\mathcal{X}$  and let  $t_n$  be an increasing sequence of positive reals such that  $\lim_{n \rightarrow \infty} t_n = \infty$ . We say that  $P$  has (starting point) cutoff at time  $t_n$  starting from  $x_n$  if

(i) For any  $c < 1$ ,

$$\lim_{n \rightarrow \infty} d_{x_n}(ct_n) = 1$$

(ii) For any  $c > 1$ ,

$$\lim_{n \rightarrow \infty} d_{x_n}(ct_n) = 0$$

Intuitively, if  $P$  has cutoff at time  $t_n$  starting from  $x_n$ , it means that for sufficiently large  $n$ , if we start a Markov chain with Markov kernel  $P$  at  $x_n$  then its distance to stationarity will be close to one for any time before  $t_n$  and close to zero at any time after  $t_n$ .

We can now state our main theorem which says that starting point cutoff is equivalent to concentration of hitting times on regular sets.

**Theorem 2.6.** Let  $x_n$  be a sequence of starting points in  $\mathcal{X}$  and let  $t_n$  be an increasing sequence of positive reals such that  $\lim_{n \rightarrow \infty} t_n = \infty$ . Let  $S$  be a regular set. Then  $P$  has cutoff at time  $t_n$  starting from  $x_n$  if and only if  $\tau_S$  concentrates at time  $t_n$ .

Using Remark 2.3, Theorem 2.6 follows immediately from Proposition 4.1 and Proposition 4.2 below.

### 3 Lemmas for a General State Space

In this section we prove two lemmas which allow us to extend the ideas from the proof of Theorem 4.1 in [7] to the general state space setting.

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**Lemma 3.1.** For any  $S \in \mathcal{B}^+$  and  $\varepsilon \in (0, 1)$  there exists  $B \in \mathcal{B}$  and  $D > 0$  such that  $\pi(B) > 1 - \varepsilon$  and  $\sup_{x \in B} \mathbb{P}_x(\tau_S \geq D/\varepsilon) \leq \varepsilon$ .

*Proof.* Since  $P$  is Harris recurrent, for all  $x \in \mathcal{X}$ ,  $\mathbb{E}_x[\tau_S] < \infty$ . Thus if we define for each  $d \in \mathbb{N}$ ,  $A_d = \{x \in \mathcal{X} : \mathbb{E}_x[\tau_S] \leq d\}$ ,  $A_d$  is an increasing sequence of measurable sets whose union is all of  $\mathcal{X}$ . Therefore by continuity of measure there exists  $N$  sufficiently large such that  $\pi(A_D) > 1 - \varepsilon$  and

$$\begin{aligned} \sup_{x \in A_D} \mathbb{P}(\tau_S \geq D/\varepsilon) &\leq (\varepsilon/D) \sup_{x \in A_D} \mathbb{E}_x[\tau_S] \quad \text{by Markov's inequality} \\ &\leq \varepsilon \quad \text{by definition of } A_D \end{aligned}$$

Therefore  $B = A_D$  is the desired set.  $\square$

**Lemma 3.2.** For any regular set  $S$ ,  $\lim_{n \rightarrow \infty} d_x(t) = 0$  uniformly on  $S$ .

*Proof.* Since  $\mathcal{B}$  is countably generated, for each fixed  $t \in \mathbb{N}$ ,  $d_x(t) : \mathcal{X} \rightarrow \mathbb{R}$  is measurable (see the Appendix of [9]). Therefore since for any  $x \in \mathcal{X}$ ,  $\lim_{t \rightarrow \infty} d_x(t) = 0$  by Egoroff's theorem (Theorem 2.33 in [4]) there exists  $B \in \mathcal{B}^+$  such that  $d_x(t)$  converges to 0 uniformly on  $B$ . Since  $S$  is a regular set there exists  $M$  such that for all  $x \in S$ ,  $\mathbb{E}_x[\tau_B] \leq M$ .

Let  $\varepsilon \in (0, 1)$ . If we define  $T_0 := \lceil \frac{M}{\varepsilon} \rceil$ , then for all  $x \in S$  and  $t \geq T_0$

$$\begin{aligned} \mathbb{P}_x(\tau_B > T_0) &\leq \frac{\mathbb{E}_x[\tau_B]}{(M/\varepsilon)} \quad \text{by Markov's inequality} \\ &\leq \varepsilon \end{aligned}$$

Since  $d_x(t)$  converges to 0 uniformly on  $B$  there exists  $T_1$  sufficiently large such that for all  $x \in B$  and  $t \geq T_1$ ,  $d_x(t) \leq \varepsilon$ .

Define  $T := T_0 + T_1$ . Then for any  $x \in S$  and  $A \in \mathcal{B}$ , if  $t \geq T$  then

$$\begin{aligned} &|P^t(x, A) - \pi(A)| \\ &= \left| \sum_{k=0}^{\infty} [\mathbb{P}_x(X_t \in A | \tau_B = k) - \pi(A)] \mathbb{P}_x(\tau_B = k) \right| \\ &\leq \sum_{k=0}^{\infty} |\mathbb{P}_x(X_t \in A | \tau_B = k) - \pi(A)| \mathbb{P}_x(\tau_B = k) \\ &\leq \mathbb{P}_x(\tau_B > T_0) + \sum_{k=0}^{T_0} |\mathbb{P}_x(X_t \in A | \tau_B = k) - \pi(A)| \mathbb{P}_x(\tau_B = k) \\ &\leq \varepsilon + \sum_{k=0}^{T_0} \sup_{y \in B} |P^{t-k}(y, A) - \pi(A)| \mathbb{P}_x(\tau_B = k) \end{aligned}$$

by the strong Markov property

$$\begin{aligned} &\leq \varepsilon + \sum_{k=0}^{T_0} \sup_{y \in B} d_y(t-k) \mathbb{P}_x(\tau_B = k) \\ &\leq \varepsilon + \sum_{k=0}^{T_0} \varepsilon \mathbb{P}_x(\tau_B = k) \quad \text{since } t-k \geq T_1 \\ &= \varepsilon + \varepsilon \mathbb{P}_x(\tau_B \leq T_0) \\ &\leq 2\varepsilon \end{aligned}$$

Therefore (since  $A$  was arbitrary) for any  $x \in S$  and  $t \geq T$ ,

$$d_x(t) \leq 2\varepsilon$$

Since  $\varepsilon$  was arbitrary this proves

$$\lim_{t \rightarrow \infty} d_x(t) = 0$$

uniformly on  $S$ . □

## 4 Proof of Equivalence

As stated in Section 2, Theorem 2.6 follows from Proposition 4.1 and Proposition 4.2 which are stated and proved in this section. It is worth noting however that these propositions prove something slightly stronger, as one direction of the equivalence in each proposition holds regardless of whether the set  $S$  is regular (as long as it is in  $\mathcal{B}^+$ ). The proof of both propositions is based heavily on the proof of Theorem 4.1 in [7], but uses the lemmas proved in Section 3 to extend it to a general state space.

**Proposition 4.1.** *Let  $x_n$  be a sequence in  $\mathcal{X}$  and let  $t_n$  be an increasing sequence of positive reals such that  $\lim_{n \rightarrow \infty} t_n = \infty$ . Let  $S \in \mathcal{B}^+$  and consider the following conditions:*

(i) For any  $c < 1$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < ct_n) = 0$$

(ii) For any  $c < 1$ ,

$$\lim_{n \rightarrow \infty} d_{x_n}(ct_n) = 1$$

If (i) holds then (ii) holds and if  $S$  is a regular set then (i) and (ii) are equivalent.

*Proof.* We first prove (i) implies (ii). Suppose (i) holds. Let  $c < 1$ . Let  $\varepsilon \in (0, 1)$ . By Lemma 3.1 there exists a set  $B \in \mathcal{B}$  and  $D > 0$  such that  $\pi(B) > 1 - \varepsilon$  and  $\sup_{x \in S} \mathbb{P}_x(\tau_S > D/\varepsilon) \leq \varepsilon$ . Since  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $c < 1$  we can choose  $N_0$  sufficiently large such that for all  $n \geq N_0$ ,  $D/\varepsilon < \frac{1-c}{2}t_n$ . Since  $\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < \frac{c+1}{2}t_n) = 0$  we can choose  $N_1$  sufficiently large such that for all  $n \geq N_1$ ,  $\mathbb{P}_{x_n}(\tau_S < \frac{c+1}{2}t_n) \leq \varepsilon$ . Define  $N := \max\{N_0, N_1\}$ . Then for all  $n \geq N$

$$\begin{aligned} & P^{\lfloor ct_n \rfloor}(x_n, B) \\ & \leq \mathbb{P}_{x_n}(\tau_B \leq ct_n) \\ & = \mathbb{P}_{x_n}(\tau_S \leq (\tau_S - \tau_B) + ct_n) \\ & \leq \mathbb{P}_{x_n}(\tau_S \leq D/\varepsilon + ct_n) + \mathbb{P}_{x_n}(\tau_S - \tau_B \geq D/\varepsilon) \quad \text{by a union bound} \\ & \leq \mathbb{P}_{x_n}(\tau_S \leq D/\varepsilon + ct_n) + \sup_{x \in B} \mathbb{P}_x(\tau_S \geq D/\varepsilon) \quad \text{by the strong Markov property} \\ & \leq \mathbb{P}_{x_n}\left(\tau_S \leq \frac{c+1}{2}t_n\right) + \varepsilon \quad \text{since } n \geq N_0 \text{ and by the definition of } B \\ & \leq \varepsilon + \varepsilon \quad \text{since } n \geq N_1 \\ & = 2\varepsilon \end{aligned}$$

therefore for all  $n \geq N$

$$d_{x_n}(ct_n) \geq \pi(B) - P^{\lfloor ct_n \rfloor}(x_n, B) \geq 1 - 3\varepsilon$$

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Since  $\varepsilon$  was arbitrary this proves

$$\lim_{n \rightarrow \infty} d_{x_n}(ct_n) = 1$$

Now we prove (ii) implies (i) when  $S$  is a regular set. Suppose  $S$  is a regular set and suppose (ii) holds. Let  $c < 1$ . Let  $\varepsilon \in (0, 1/2)$ . Since  $\lim_{n \rightarrow \infty} d_{x_n}(\frac{c+1}{2}t_n) = 1$  there exists  $N_0$  sufficiently large such that for each  $n \geq N_0$ , there exists  $A_n \in \mathcal{B}$  such that  $\pi(A_n) > 1 - \varepsilon$  and  $P^{\lfloor \frac{c+1}{2}t_n \rfloor}(x_n, A_n) < \varepsilon$ . By Lemma 3.2 we can choose  $T_0$  sufficiently large such that for any  $x \in S$  and  $t \geq T_0$ ,  $d_x(t) < 1/4$ . In particular this implies for any  $n \geq N_0$  and  $t \geq T_0$  that  $\inf_{x \in S} P^t(x, A_n) > \pi(A_n) - 1/4 > 1/4$ . Furthermore since  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $c < 1$  we can choose  $N_1$  sufficiently large such that for all  $n \geq N_1$  and  $t \leq \lfloor ct_n \rfloor$  we have  $\lfloor \frac{c+1}{2}t_n \rfloor - t \geq T_0$ . Define  $N = \max\{N_0, N_1\}$ . Then for any  $n \geq N$  we have

$$\begin{aligned} & \mathbb{P}_{x_n}(\tau_S < ct_n) \\ & \leq \sum_{t=0}^{\lfloor ct_n \rfloor} \mathbb{P}_{x_n}(\tau_S = t) \\ & \leq 4 \left( \sum_{t=0}^{\lfloor ct_n \rfloor} \inf_{x \in S} P^{\lfloor \frac{c+1}{2}t_n \rfloor - t}(x, A_n) \mathbb{P}_{x_n}(\tau_S = t) \right) \quad \text{since } n \geq N_0 \text{ and } n \geq N_1 \\ & \leq 4 \left( \sum_{t=0}^{\lfloor ct_n \rfloor} \mathbb{P}_{x_n}(X_{\lfloor \frac{c+1}{2}t_n \rfloor} \in A_n \mid \tau_S = t) \mathbb{P}_{x_n}(\tau_S = t) \right) \quad \text{by the strong Markov property} \\ & = 4\mathbb{P}_{x_n}(X_{\lfloor \frac{c+1}{2}t_n \rfloor} \in A_n, \tau_S \leq \lfloor ct_n \rfloor) \\ & \leq 4P^{\lfloor \frac{c+1}{2}t_n \rfloor}(x_n, A_n) \\ & \leq 4\varepsilon \end{aligned}$$

Since  $\varepsilon$  was arbitrary, this proves

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < ct_n) = 0$$

□

**Proposition 4.2.** *Let  $x_n$  be a sequence in  $\mathcal{X}$  and let  $t_n$  be an increasing sequence of positive reals such that  $\lim_{n \rightarrow \infty} t_n = \infty$ . Let  $S \in \mathcal{B}^+$  and consider the following conditions:*

(i) For any  $c > 1$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < ct_n) = 1$$

(ii) For any  $c > 1$ ,

$$\lim_{n \rightarrow \infty} d_{x_n}(ct_n) = 0$$

If (ii) holds then (i) holds and if  $S$  is a regular set then (ii) and (i) are equivalent.

*Proof.* We first prove (ii) implies (i). Suppose (i) holds. Let  $c > 1$ . Let  $\varepsilon \in (0, 1/2)$ . By Lemma 3.1 there exists a set  $B \in \mathcal{B}$  and  $D > 0$  such that  $\pi(B) > 1 - \varepsilon$  and  $\sup_{x \in S} \mathbb{P}_x(\tau_S > D/\varepsilon) \leq \varepsilon$ . Since  $\lim_{n \rightarrow \infty} d_{x_n}(\frac{c+1}{2}t_n) = 0$ , we can choose  $N_0$  sufficiently large such that for all  $n \geq N_0$ ,  $d_{x_n}(\frac{c+1}{2}t_n) < \varepsilon$ . In particular this implies that for all  $n \geq N_0$ ,

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$P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, B) \geq \pi(B) - \varepsilon > 1 - 2\varepsilon$ . Since  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $c > 1$ , we can choose  $N_1$  sufficiently large such that for all  $n \geq N_1$  we have  $ct_n - \lfloor \frac{c+1}{2} t_n \rfloor > D/\varepsilon$ . Define  $N = \max\{N_0, N_1\}$ . Then for any  $n \geq N$ ,

$$\begin{aligned}
& \mathbb{P}_{x_n}(\tau_S < ct_n) \\
& \geq \mathbb{P}_{x_n}(\tau_S < ct_n, X_{\lfloor \frac{c+1}{2} t_n \rfloor} \in B) \\
& = \int_B \mathbb{P}_{x_n}(\tau_S < ct_n \mid X_{\lfloor \frac{c+1}{2} t_n \rfloor} = y) P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, dy) \\
& = \int_B P_y \left( \tau_S < ct_n - \left\lfloor \frac{c+1}{2} t_n \right\rfloor \right) P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, dy) \quad \text{by the strong Markov property} \\
& = \int_B \left[ 1 - P_y \left( \tau_S \geq ct_n - \left\lfloor \frac{c+1}{2} t_n \right\rfloor \right) \right] P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, dy) \\
& \geq \int_B [1 - P_y(\tau_S > D/\varepsilon)] P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, dy) \quad \text{since } n \geq N_1 \\
& \geq \int_B \left[ 1 - \sup_{x \in B} P_x(\tau_S > D/\varepsilon) \right] P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, dy) \\
& \geq \int_B (1 - \varepsilon) P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, dy) \quad \text{by definition of } B \\
& = (1 - \varepsilon) P^{\lfloor \frac{c+1}{2} t_n \rfloor}(x_n, B) \\
& \geq (1 - \varepsilon)(1 - 2\varepsilon) \quad \text{since } n \geq N_0
\end{aligned}$$

Since  $\varepsilon$  was arbitrary this proves

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S < ct_n) = 1$$

Now we prove (i) implies (ii) when  $S$  is a regular set. Suppose  $S$  is a regular set and suppose (i) holds. Let  $c > 1$ . Let  $\varepsilon \in (0, 1)$ . Since  $\lim_{n \rightarrow \infty} \mathbb{P}_{x_n}(\tau_S \leq \frac{c+1}{2} t_n) = 1$ , we can choose  $N_0$  sufficiently large such that for all  $n \geq N_0$ ,  $\mathbb{P}_{x_n}(\tau_S \leq \frac{c+1}{2} t_n) > 1 - \varepsilon$ . By Lemma 3.2 we can choose  $T_0$  sufficiently large such that for any  $x \in S$  and  $t \geq T_0$ ,  $d_x(t) \leq \varepsilon$ . Furthermore since  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $c > 1$  we can choose  $N_1$  sufficiently large such that for all  $n \geq N_1$  and  $t \leq \lfloor \frac{c+1}{2} t_n \rfloor$  we have  $\lfloor ct_n \rfloor - t \geq T_0$ . Define  $N = \max\{N_0, N_1\}$ . Let  $A \in \mathcal{B}$ . Then for all  $n \geq N$  we have

$$\begin{aligned}
& |P^{\lfloor ct_n \rfloor}(x_n, A) - \pi(A)| \\
& = \left| \sum_{t=0}^{\infty} [\mathbb{P}_{x_n}(X_{\lfloor ct_n \rfloor} \in A \mid \tau_S = t) - \pi(A)] \mathbb{P}_{x_n}(\tau_S = t) \right| \\
& \leq \sum_{t=0}^{\infty} |\mathbb{P}_{x_n}(X_{\lfloor ct_n \rfloor} \in A \mid \tau_S = t) - \pi(A)| \mathbb{P}_{x_n}(\tau_S = t) \\
& \leq \mathbb{P}_{x_n} \left( \tau_S > \frac{c+1}{2} t_n \right) + \sum_{t=0}^{\lfloor \frac{c+1}{2} t_n \rfloor} |\mathbb{P}_{x_n}(X_{\lfloor ct_n \rfloor} \in A \mid \tau_S = t) - \pi(A)| \mathbb{P}_{x_n}(\tau_S = t) \\
& \leq \varepsilon + \sum_{t=0}^{\lfloor \frac{c+1}{2} t_n \rfloor} \sup_{x \in S} |P^{\lfloor ct_n \rfloor - t}(x, A) - \pi(A)| \mathbb{P}_{x_n}(\tau_S = t)
\end{aligned}$$

since  $n \geq N_0$  and by the strong Markov property

$$\begin{aligned}
 &\leq \varepsilon + \sum_{t=0}^{\lfloor \frac{c+1}{2} t_n \rfloor} \sup_{x \in S} d_x(\lfloor ct_n \rfloor - t) \mathbb{P}_{x_n}(\tau_S = t) \\
 &\leq \varepsilon + \sum_{t=0}^{\lfloor \frac{c+1}{2} t_n \rfloor} \varepsilon \mathbb{P}_{x_n}(\tau_S = t) \quad \text{since } n \geq N_1 \\
 &\leq \varepsilon + \varepsilon \mathbb{P}_{x_n} \left( \tau_S \leq \left\lfloor \frac{c+1}{2} t_n \right\rfloor \right) \\
 &\leq 2\varepsilon
 \end{aligned}$$

Therefore (since  $A$  was arbitrary) for any  $n \geq N$

$$d_{x_n}(ct_n) \leq 2\varepsilon$$

Since  $\varepsilon$  was arbitrary

$$\lim_{n \rightarrow \infty} d_{x_n}(ct_n) = 0$$

□

## 5 A Sufficient Condition for Cutoff

Theorem 2.6 reduces the problem of exhibiting a starting point cutoff phenomenon for a general state space Markov chain to the problem of showing a concentration of hitting times to some regular set  $S$ . In general this may be just as difficult to show, but in some cases this may be much easier. In this section we use Theorem 2.6 to establish a sufficient condition for starting point cutoff. We will use this result in Section 6 to show that any regular random walk on the half-line with bounded steps exhibits the starting point cutoff phenomenon.

**Proposition 5.1.** *Suppose there exists a constant  $C$  such that for all  $x \in \mathcal{X} \setminus S$*

$$|\mathbb{E}_{X_1}[\tau_S] - \mathbb{E}_x[\tau_S]| \leq C \quad \text{almost surely}$$

*Then for any sequence of starting points  $x_n$  such that  $\lim_{n \rightarrow \infty} \mathbb{E}_{x_n}[\tau_S] = \infty$  we have that  $\tau_S$  (starting from  $x_n$ ) concentrates at time  $t_n = \mathbb{E}_{x_n}[\tau_S]$ .*

In order to prove the Proposition 5.1 (a qualitative result) we first prove a quantitative concentration inequality. The proof of this lemma is based on a well-known technique for establishing concentration inequalities for hitting times using the classical Azuma's inequality and the fact that the shifted expected hitting times form a martingale (for example see [6]).

**Lemma 5.2.** *Suppose there exists a constant  $C$  such that for all  $x \in \mathcal{X} \setminus S$*

$$|\mathbb{E}_{X_1}[\tau_S] - \mathbb{E}_x[\tau_S]| \leq C \quad \text{almost surely}$$

*Then for any  $x \in \mathcal{X}$  and  $\varepsilon > 0$ ,*

$$\mathbb{P}_x(|\tau_S - \mathbb{E}_x[\tau_S]| \geq \varepsilon \mathbb{E}_x[\tau_S]) \leq 2 \exp\left(\frac{-2\varepsilon^2 \mathbb{E}_x[\tau_S]}{(1 + \varepsilon)C^2}\right)$$

To prove this lemma we state (without proof) the version of the Azuma's inequality we will use (see Theorem 5.1 in [2]).



Starting point cutoff on a general state space

**Theorem 5.3.** Suppose  $\{M_t\}_{t=0}^T$  is a martingale with respect to a filtration  $\{\mathcal{F}_t\}_{t=0}^T$  and there exists  $C \geq 0$  such that for each  $t \geq 1$ ,

$$|M_t - M_{t-1}| \leq C \quad \text{almost surely}$$

Then for any  $\varepsilon > 0$ ,

$$\mathbb{P}(|M_T - M_0| \geq \varepsilon) \leq 2 \exp\left(\frac{-2\varepsilon^2}{TC^2}\right)$$

*Proof of Theorem 5.2.* Let  $X_0 = x$ . Fix  $\varepsilon > 0$ . Let  $T = \lfloor (1 + \varepsilon)\mathbb{E}_{x_n}[\tau_S] \rfloor$ . For  $t$  from 0 to  $T$  define

$$M_t = \begin{cases} \mathbb{E}_{X_t}[\tau_S] + t & \text{when } t < \tau_S \\ \tau_S & \text{when } t \geq \tau_S \end{cases}$$

It is easy to check that  $M_t$  is a martingale with respect to the filtration  $\mathcal{F}_t = \sigma(X_0, \dots, X_t)$ . Then for any  $t$  from 1 to  $T$ , either  $X_{t-1} \in S$  in which case  $|M_t - M_{t-1}| = 0$ , or  $X_{t-1} \in \mathcal{X} \setminus S$  in which case  $|M_t - M_{t-1}| = |\mathbb{E}_{X_t}[\tau_S] - \mathbb{E}_{X_{t-1}}[\tau_S] + 1| \leq C + 1$  almost surely. Therefore  $|M_t - M_{t-1}| \leq C + 1$  almost surely. Thus by Theorem 5.3,

$$\mathbb{P}(|M_T - M_0| \geq \varepsilon \mathbb{E}_x[\tau_S]) \leq 2 \exp\left(\frac{-2\varepsilon^2 \mathbb{E}_x[\tau_S]}{(1 + \varepsilon)C^2}\right)$$

We will finish the proof by showing that the event  $|\tau_S - \mathbb{E}_x[\tau_S]| \geq \varepsilon \mathbb{E}_x[\tau_S]$  implies the event  $|M_T - M_0| \geq \varepsilon \mathbb{E}_x[\tau_S]$  and therefore

$$\begin{aligned} \mathbb{P}_x(|\tau_S - \mathbb{E}_x[\tau_S]| \geq \varepsilon \mathbb{E}_x[\tau_S]) &\leq \mathbb{P}(|M_T - M_0| \geq \varepsilon \mathbb{E}_x[\tau_S]) \\ &\leq 2 \exp\left(\frac{-2\varepsilon^2 \mathbb{E}_x[\tau_S]}{(1 + \varepsilon)C^2}\right) \end{aligned}$$

Suppose first  $\tau_S - \mathbb{E}_x[\tau_S] > \varepsilon \mathbb{E}_x[\tau_S]$ . Then  $\tau_S > (1 + \varepsilon)\mathbb{E}_x[\tau_S] \geq T$ . In particular this implies  $M_T = \mathbb{E}_{X_T}[\tau_S] + \lfloor (1 + \varepsilon)\mathbb{E}_x[\tau_S] \rfloor$  and since  $M_0 = \mathbb{E}_x[\tau_S]$  we have that  $|M_T - M_0| \geq \mathbb{E}_{X_T}[\tau_S] + \lfloor (1 + \varepsilon)\mathbb{E}_x[\tau_S] \rfloor - \mathbb{E}_x[\tau_S] \geq (\mathbb{E}_{X_T}[\tau_S] - 1) + \varepsilon \mathbb{E}_x[\tau_S] \geq \varepsilon \mathbb{E}_x[\tau_S]$ . In the other case suppose  $\mathbb{E}_x[\tau_S] - \tau_S \geq \varepsilon \mathbb{E}_x[\tau_S]$ . Then  $T \geq (1 - \varepsilon)\mathbb{E}_x[\tau_S] > \tau_S$ . In particular this implies that  $M_T = \tau_S$  and since  $M_0 = \mathbb{E}_x[\tau_S]$  we have that  $|M_T - M_0| = |\tau_S - \mathbb{E}_x[\tau_S]| = \mathbb{E}_x[\tau_S] - \tau_S \geq \varepsilon \mathbb{E}_x[\tau_S]$ .  $\square$

*Proof of Proposition 5.1.* By Theorem 2.6 it suffices to show that  $\tau_S$  concentrates at times  $t_n = \mathbb{E}_{x_n}[\tau_S]$ . Let  $\varepsilon > 0$ . For each  $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}_{x_n} \left( \left| \frac{\tau_S}{\mathbb{E}_{x_n}[\tau_S]} - 1 \right| \geq \varepsilon \right) &= \mathbb{P}_x(|\tau_S - \mathbb{E}_x[\tau_S]| \geq \varepsilon \mathbb{E}_x[\tau_S]) \\ &\leq 2 \exp\left(\frac{-2\varepsilon^2 \mathbb{E}_x[\tau_S]}{(1 + \varepsilon)C^2}\right) \quad \text{by Lemma 5.2} \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  shows that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n} \left( \left| \frac{\tau_S}{\mathbb{E}_{x_n}[\tau_S]} - 1 \right| \geq \varepsilon \right) = 0$$

and therefore

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_n} \left( \left| \frac{\tau_S}{\mathbb{E}_{x_n}[\tau_S]} - 1 \right| \leq \varepsilon \right) = 1$$

$\square$

## 6 Random Walks on the Half Line

In this section we apply Proposition 5.1 to show that all random walks on the half-line with bounded step size exhibit the starting point cutoff phenomenon. Let  $\{W_t\}_{t \geq 1}$  be a sequence of i.i.d real-valued random variables and let  $x \in [0, \infty)$ . We define a random walk  $X_t$  on the half-line  $[0, \infty)$  recursively by setting:

$$\begin{aligned} X_0 &= x, \\ X_t &= [X_{t-1} + W_t]^+ \quad \text{for } t \geq 1 \end{aligned}$$

where  $[x]^+ = \max\{x, 0\}$ . We will also assume that  $\mathbb{E}[W_t] = \mu < 0$ . It is shown in Proposition 11.4.1 of [8] that this Markov chain is regular if and only if  $\mu < 0$ , and in this case all compact sets are regular sets. We will consider in particular the regular set  $S = \{0\}$ . We say that  $X_t$  is a random walk on the half-line with bounded steps if there exists  $C > 0$  such that  $|W_t| \leq C$  almost surely.

**Theorem 6.1.** *Suppose  $X_t$  is a random-walk on the half-line with bounded steps. Then for any sequence of starting points  $x_n \in [0, \infty)$ , if  $\lim_{n \rightarrow \infty} x_n = \infty$  then  $X_t$  exhibits starting point cutoff at time  $t_n = \mathbb{E}_{x_n}[\tau_{\{0\}}]$  starting from  $x_n$ .*

*Proof.* Fix  $x \in \mathcal{X}$  and let  $X_0 = x$ . Let  $S_t = \sum_{i=1}^t W_i$ . Note that

$$\tau_{\{0\}} = \inf \{t \geq 0 \mid X_t = 0\} = \inf \{t \geq 0 \mid S_t \leq -x\}$$

Therefore by Wald's equation (see Theorem 2.6.2 in [3]) we have

$$\mathbb{E}_x[\tau_{\{0\}}] = \frac{\mathbb{E}[S_{\tau_{\{0\}}}]}{\mu}$$

but since

$$-(C + x) \leq W_{\tau_{\{0\}}} - x \leq W_{\tau_{\{0\}}} + S_{(\tau_{\{0\}}-1)} = S_{\tau_{\{0\}}} \leq -x \quad \text{almost surely:}$$

we have that

$$-(C + x) \leq \mathbb{E}[S_{\tau_{\{0\}}}] \leq -x$$

and therefore

$$-\frac{(C + x)}{\mu} \leq \mathbb{E}_x[\tau_{\{0\}}] \leq -\frac{x}{\mu}$$

thus

$$|(\mathbb{E}_{X_1}[\tau_{\{0\}}] - \mathbb{E}_x[\tau_{\{0\}}])| \leq \frac{(|X_1 - x| + C)}{\mu} = \frac{(|W_1| + C)}{\mu} \leq \frac{2C}{\mu} \quad \text{almost surely}$$

Therefore by Proposition 5.1,  $X_t$  exhibits starting point cutoff at time  $t_n = \mathbb{E}_{x_n}[\tau_{\{0\}}]$  starting from  $x_n$ .  $\square$

## 7 An Example Without Starting Point Cutoff

In this section we construct an example of a Markov chain with a sequence of starting points  $x_n$  that diverges to infinity (in the sense that the expected hitting time of a fixed regular set from those starting points diverges to infinity) where the Markov chain does not exhibit the starting point cutoff phenomenon from those starting points (meaning

## Starting point cutoff on a general state space

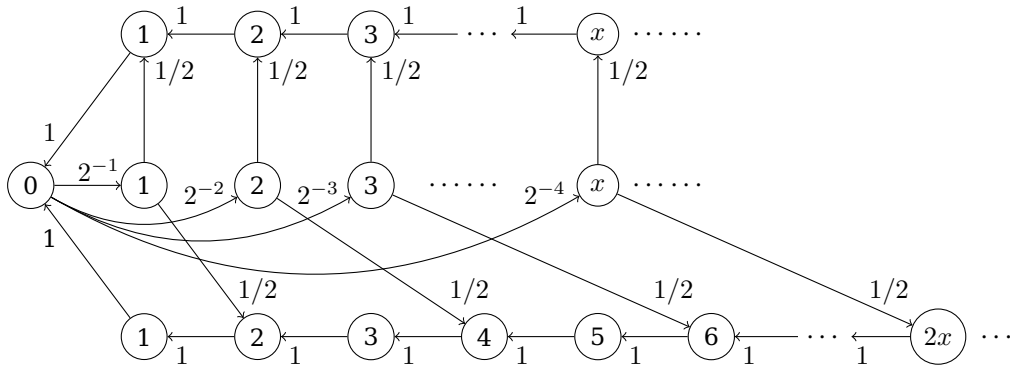


Figure 1: Example Chain

precisely that there does not exist a sequence of positive reals  $t_n$  for which the Markov kernel associated with the chain has starting point cutoff at time  $t_n$  starting from  $x_n$ ).

Consider the Markov chain depicted in Figure 1. Its state space is three disjoint copies of  $\mathbb{N}$  glued together at 0. When at 0 the Markov chain jumps to a state in the middle row with a Geometric(1/2) distribution. When at a non-zero state  $x$  in the middle row the Markov chain jumps to  $x$  in the top row with probability 1/2 or jumps to  $2x$  in the bottom row with probability 1/2. Once in the top or bottom row the Markov chain jumps deterministically down in the same row one step at a time until it reaches 0. It is easy to see that this is an aperiodic regular Markov chain and  $S = \{0\}$  is a regular set. Now for any increasing sequence of starting points  $x_n$  in the middle row we have that  $\mathbb{P}_{x_n}(\tau_S = x_n) = \frac{1}{2}$  and  $\mathbb{P}_{x_n}(\tau_S = 2x_n) = \frac{1}{2}$  so it is clear that  $\lim_{n \rightarrow \infty} \mathbb{E}_{x_n}[\tau_S] = \infty$  and  $\tau_S$  starting from  $x_n$  cannot concentrate at any (deterministic) time  $t_n$  for any sequence of positive reals  $t_n$ . Therefore by Theorem 2.6 for any increasing sequence of starting points  $x_n$  in the middle row, the Markov chain does not exhibit the starting point cutoff phenomenon.

## 8 Open Problems

In this section we state and discuss three open problems inspired by the results of this paper. The first proposes a possible extension of the theory, and the latter two propose extensions of the application.

**Question 8.1.** Can one characterize the starting point cutoff phenomenon for a sequence of (possibly distinct) regular Markov chains by a concentration of hitting times (generalizing both Theorem 2.6 and Theorem 1 of [5])?

Theorem 1 of [5] shows that for a sequence of finite irreducible reversible Markov chains satisfying the product condition, cutoff starting from a sequence of starting distributions can be characterized by concentration of hitting times for a sequence of sets “worst in expectation”. While this result only holds for reversible chains and does not apply directly to the general state space setting of Theorem 2.6, one could imagine a generalization that applies to sequences of regular Markov chains (on a general state space). In the case where the sequence of Markov chains is constant this should reduce to Theorem 2.6. Since the main tool of [5], Starr’s maximal inequality [10], applies on a general probability space, we believe it is likely that Theorem 1 of [5] can be extended to sequences of reversible regular Markov chains on a general state space in a similar way that this paper extends the results of [7] using ideas from [8].

**Question 8.2.** Could Proposition 5.1 be used to establish a cutoff phenomenon for other

general classes of chains?

One can think of many examples where the hypothesis of Proposition 5.1 likely holds but is hard to verify. This is because there does not seem to be any good tools for computing or bounding  $\mathbb{E}_x[\tau_S]$  in general. There may however be other interesting general classes of chains where bounds on this expectation could be derived and the hypothesis of Proposition 5.1 could be shown to hold.

**Question 8.3.** What are the necessary and sufficient conditions on the (possibly unbounded) distribution of  $W_t$  for the random walk on the half-line  $X_t$  to exhibit starting point cutoff?

We showed in Theorem 6.1 that bounded steps is a sufficient condition for random walks on the half-line to exhibit the starting point cutoff phenomenon, but it is easy to construct examples which show this is not necessary. It would be interesting to determine precisely under what conditions a random walk on the half-line exhibits starting point cutoff.

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